Risk Aversion Impact on Investment Strategy Performance: A Multi Agent-Based Analysis

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Abstract In order to supply an additional evidence on the effect of individual investors preferences on their portfolio dynamics from the wealth and risk adjusted return point of view, we construct an agent-based multi-asset model. We populate the artificial market with heterogeneous mean-variance traders with quadratic utility function. We compare the relative performance of investment strategies differ on their risk preferences using ecological competitions, where populations of artificial investors co-evolve. Our findings show that the higher relative risk aversion helps the agents survive in a long-range time frame in the competitions for higher wealth or Sharpe ratio of constrained portfolios. However, when short-selling is allowed, the highest (as well as lowest) risk aversion does not guarantee the highest earnings. Risk lovers as well as absolute risk averters run quickly out of competitions. Only the traders with moderate level of risk aversion survive in the long run.

1 Introduction

Multi-agent simulations of financial market seek to address investment problems by providing the conditions for a controlled experiment, and thus allowing us to illustrate cause and effect relationships between initial scenario settings and market behavior. In this paper we apply this tool to shed some new light on a classical portfolio optimization problem. Markowitz [20] first formalized the portfolio optimization problem in mean-variance framework (thus the problem is known as mean-variance portfolio optimization). Since

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then, this model has been actively extended and investigated. One of the important issues is to understand how agents individuals' degree of risk aversion affect their investment decisions and outcomes. Direct empirical investigation of this question is typically quite difficult, due to the difficulty to obtain reliable statistics and to control for many environmental effects which may also affect behavior. The data obtained from simulations in artificial markets with artificial investors can be a good supplement to the real data.

We introduce agents with heterogeneous risk preference into the artificial stock market in order to address the question whether investors' survivability depends on their risk preferences. One of the critics of agent-based models in finance is that the computational methods produce only examples, which can not explain the relative importance of model parameters settings and model outputs. A few examples may not shed some new light on the investigated phenomena, but a few thousand well chosen examples can be more convincing [10]. Thus, we introduce thousand of agents trading on the market over thousands of days. Extensive computational results are presented. We compare the relative performance of investment strategies differ on their risk preferences using Ecological Competition [18], [26] where populations of artificial investors co-evolve.

2 The advantages of the proposed heterogeneous multi-agent model

Risky financial securities should generate, in equilibrium, a return higher than the one of safer investments such as Treasury Bills [21].[1] Risk preferences of investors have a direct impact on their investment decisions. A risk-averter (or conservative) investor tends to hold more Treasury Bills than a risk-lover (or aggressive) investor who will tend to invest in more risky stocks with higher expected return. Thus, risk aversion affects the portfolio composition of investors and therefore the distribution of future wealth. In other words, each trader invests his capital in a portfolio reflecting his risk-aversion.

This work is motivated by empirical studies focusing on the relation between risk aversion and wealth dynamics (see for example, [13]). Several agent-based simulations researches have also investigated this question. In fact, some simulation-based works lean towards a framework where investor's optimal decisions depend on their wealth, which is in line with the assumption of CRRA utility function, [14], [15], [16]. [7] investigate the characteristics of asset prices and wealth dynamics arising from the interaction of heterogeneous agents with CRRA utility. [15] study the effect of heterogeneity of preferences, expectation and strategies on wealth and price dynamics with CRRA and logarithmic utility functions using a microscopic simulation ap-

[1] For example, the CAPM [24] assumes a linear risk-return relationship $\mu_{P,t} = r_f + \beta_P \sigma_{i,t}$, where μ_P is a portfolio expected return, r_f the risk-free rate, β_P the portfolio beta and σ_{it} the market risk premium.

proach. [5], [4], [6] investigate relative risk aversion (later RRA) relation with wealth dynamics (CRRA utility function) and relation between RRA and survival dynamics (CRRA, CARA, Logarithmic, CAPM). They found that only the CRRA investors with RRA coefficient close to one can survive in the long-run time framework.

Most of the financial models with heterogeneous agents modeled with an expected utility function are built under the assumption that investors trade only *one risky asset* and one risk free asset ([1]) : in the present work *the number of assets varies*. Indeed, [16] show that the number of traded assets can significantly affect the output results. Additionally, equilibrium price is usually determined according to *simple market clearing mechanism* (as a response to excess demand or supply) [2]. In this research, we use a powerful platform for our simulations, denominated ATOM [25], which explicitly implements a *continuous double auction mechanism* with real market orders.

Next, most of the models in the agent-based literature consider the time period between each trade equal to one day or one year. In these models, during the interval between time t and time $t+1$ there is no trade, and the price does not change (see for instance [23]) which can be seen as an unrealistic feature potentially altering the outcomes of these models. In this research, we reproduce *years* of trading at a fine grain level *through intraday trading sessions*.

Finally, most of the models are built under the assumption of *constant proportions of agents*. In order to overcome this unconvincing element, [3], [2] introduce a new switching mechanism along which the wealth of each group of agents following the same strategy is updated from period t to $t+1$, not only as a consequence of agents' portfolio growth, but also due to the flow of agents coming from other groups. In our research we guarantee *the evolution of agents proportions* using a controlled ecological competition principle [18], [26]. Agents proportions are updated from period t to $t+1$ according to a performance indicator, such as the average wealth or the average Sharpe Ratio[2], delivered during the last trading period by each group of agents. As mentioned in [23] the strategy alone does not allow forecasting which population will prevail. Its success depends on the market conditions (other market participants).

Estimation of the Risk Aversion measure

We consider A as the risk aversion parameter indicating the pretension of agent to take a risk. Agents with $A = 0$ are the risk lovers or aggressive traders while agents with $A \geq 1$ are absolute risk averters (or conservative traders). The literature provides a large range for risk aversion estimation. For example, the lowest risk aversion measure producing a profit is found in [19] and is equal to $A = 0.3$. [9] define the possible ranges of risk aversion

[2] *Sharpe Ratio* $= \frac{\mu_p - r_f}{\sigma_p}$, where μ_p is the return of portfolio, r_f – risk free rate, σ_p – the standard deviation of portfolio's returns.

as 0.3502 and 0.9903. [8] use risk aversion value between 0.6 and 1.4. [6] define risk aversion in the range $[0.5, 5]$ with CRRA utility. [15] investigate two groups of agents with risk aversion measures equal to 0.5 and 3.5 (in a CRRA framework). Risk aversion is equal to 18 in [22], 30 in [12]. [11] define the ranges for risk aversion parameter for a quadratic utility function as $0 \to \infty$. In this research, we consider the range $[0.1, 10]$ as representative of the different levels mentioned in the literature.

3 Simulations design and results

We consider a complete securities market populated by a finite number of mean-variance traders with *heterogeneous* preferences, indexed by $i \in 1, 2, ...I$. Time in this model is discrete and indexed by $t = 0, 1, 2, ...$ There is also a finite number of assets $j \in 1, 2, ...J$. The trader i comes to the market with an initial amount of stocks $(q_{i,1}, q_{i,2}, ...q_{i,J})$, $i = 1...I$. We denote the trader $i's$ wealth at time t by $W_t^i = \sum_{j=1,J} p_{j,t} \cdot q_{j,t}^i + C_t^i$, where $p_{j,t}$ - is the current market price of asset j at time t, and $q_{j,t}^i$ - is the quantity of assets j held by trader i in t; C_t^i is an available cash held by the trader i at time t. The trader i defines the target weights of an optimal portfolio $\alpha_t^{i,*} = (\alpha_{1,t}^{1,*}, \alpha_{2,t}^{i,*}, \alpha_{j,t}^{i,*})$. Such allocation is a solution of convex quadratic programming problem [20] and it reflects agent's risk preferences or *rate of risk aversion*. Based on this information, the mean-variance trader calculates a desired quantity of stocks required for portfolio diversification.

$$q_{j,t}^{i,*} = \frac{\alpha_{j,t}^{i,*} \cdot W_t^i}{p_{j,t}} \tag{1}$$

In order to get closer to target weights, the trader issues "buy" or "sell" orders regarding all the assets presently traded in the market. If the difference between the desired amount of stocks $q_{j,t}^{i,*}$ held by the trader i at time t and the real amount $q_{j,t-1}^i$ held by this trader in $t-1$, the investor issues a bid order. If this difference is negative, the trader issues an ask order (sell). In other case, the trader holds unchanged positions.

The other important question is the definition and adjustment of the price mentioned in the investor's order. In our model, this process is defined by two following equations:

– *Bid price*

$$P_{Bidt} = P_{Bidt-1} + \beta_t \tag{2}$$

where P_{Bidt-1} is the best bid price in the order book at the moment $t-1$, β_t is a random value in the range $[1; 10]$; this means that best bid price in t will be increased by a value from 1 to 10 cents. P_{Bid0} is equal to the previous day closing price.

– *Ask price*

$$P_{Ask\,t} = P_{Ask\,t-1} + \alpha_t \tag{3}$$

where $P_{Ask\,t-1}$ is the best ask price in the order book at the moment $t-1$, α_t is the random value in the range $[1;10]$, which means that best ask price at time t will be decreased by a value from 1 to 10 cents. Similarly, $P_{Ask\,0}$ is equal to the previous day closing price.

We introduce a series of assumptions for our experiments:

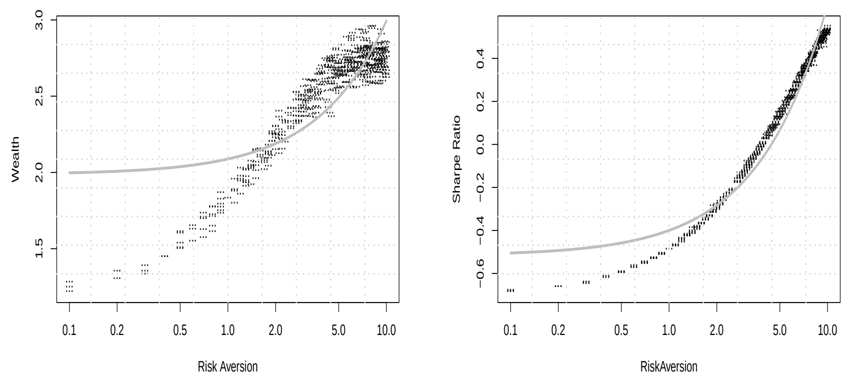
– All information concerning the underlying probability distribution of security prices as well as the current security prices are perfect information available continuously and costlessly to all the agents. All of them use the same memory span – 100 days of price history.
– Agents have an open access to the market in order to monitor their portfolio.
– They enter the market with $1000 cash and 50 units of each class of assets.
– The number of mean-variance traders (see below) is 1000.
– A day contains 700 rounds of continuous trading. Before this continuous trading session, we implement a 100 rounds opening and after, a 100 rounds closing sessions, both being ruled by a fixing mechanism (one price maximizing the exchanged volume).
– Mean-variance traders decide to rebalance their portfolio once a day.
– Risk aversion A varies from 0.1 to 10, with an increment of 0.1.

Results and Discussions

First of all, we estimate the performance of trading strategies based on end-of-the-period values like in most of the models dealing with similar research question. Then, we put the agents in competition such that the populations of investors co-evolve: agents change their strategy between the trading periods based on their historical performance. Finally, we compare the results.

We run 1500 days of trading (which corresponds to 5-year or 15 trading periods, 100 days each).For the first trading period (100 days) we provide the initial statistics for the traded assets to the mean-variance traders. During the next periods, agents calculate assets statistics themselves, based on the generated price series. The traders do not change their risk preferences and their trading strategies between periods (in ecological competition framework this constraint will be relaxed). We run 100 simulations with different initial assets statistics. We also test short-selling and long-only cases. We begin by discussing the *3-asset case*.

In figure 1(a) we show the relation between agents' risk preferences and their wealth distribution. On the horizontal axis we set out the different initial parameters – risk aversion between 0.1 and 10, with 0.1 as increment in log-scale. The vertical axis shows the final wealth corresponding to these different



(a) Wealth Distribution for Agents with Different Risk Aversions. *Regression coefficient*= 0.10065

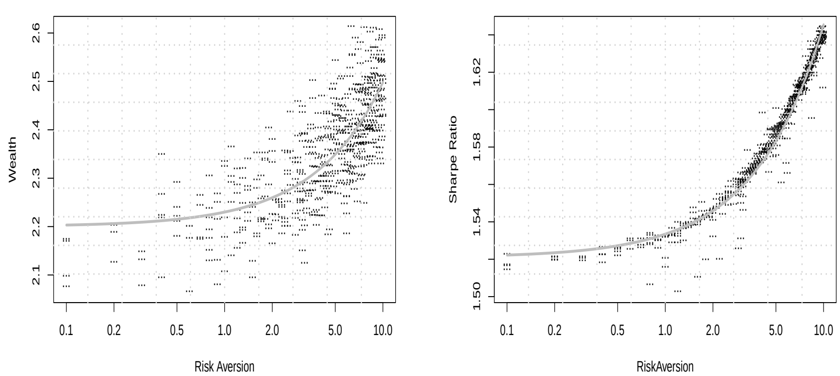
(b) Sharpe Ratio Distribution for Agents with Different Risk Aversions. *Regression coefficient* = 0.11678

Fig. 1 3-asset long-only case. Each point is the averaged value of 100 simulations. X axis is in log-scale

initial parameters. A great difference between the wealth distribution and its linear regression fitting (a gray solid line) indicates that the wealth increases sharply for agents with risk aversion from 0.1 to 3.5. Thereafter, it increases smoothly. This behavior can be explained by the composition of the optimal portfolio. [11] provide guidance regarding the significance of the changes in risk aversion for optimal portfolio composition. Agents with $A > 4$ are very risk averse and prefer portfolio with low variance. If a degree of risk aversion A is superior of 4, the portfolio composition does not vary even for large changes in A. Range $2 \leq A \leq 4$ yields moderately risky portfolios with a modest degree of change in the optimal portfolio with changes in parameter A. The range $0 \leq A \leq 2$ yields risky portfolios and there are dramatic changes in the target weights for even small changes in A.

We also investigate risk-adjusted reward to volatility of individual portfolios, also known as the Sharpe ratio. Observing the Sharpe ratio dynamic over different risk aversion frameworks (see figure 1(b)), we get similar results as [6]. Even if highly-risk-averse agents choose assets with low risk and low return, they earn a higher Sharpe ratio and a higher final wealth. This effect can be explained by the mathematical properties of the efficient frontier. The first derivative of portfolio return μ_p with respect to portfolio risk σ_p indicates that a big values of A (the minimal variance portfolio) corresponds to a big slope on the efficient frontier. Hence, conservative investors get significant increasing of portfolio return by undertaking a small amount of extra risk. The slope becomes smaller when A decreases. The second derivative of μ_p with respect to σ_p is negative, which means that the efficient frontier is concave. For large values of A, the second derivative has a large negative magnitude, so the slope is sharply decreasing. With $A \to 0$ the slope decreases much more slowly. Contrary to [5] and [6], in our simulations, less risk-averse agents ($A < 1$) do not run out of the market, even if, on average, they obtain a lower gain than risk averters ($A > 1$). If the number of assets remains relatively small and short selling is allowed, the Sharpe Ratio distribution in relation to risk aversion is close to that received with long-only constraint. Thus, the 3-asset short-selling case is not considered in current work.

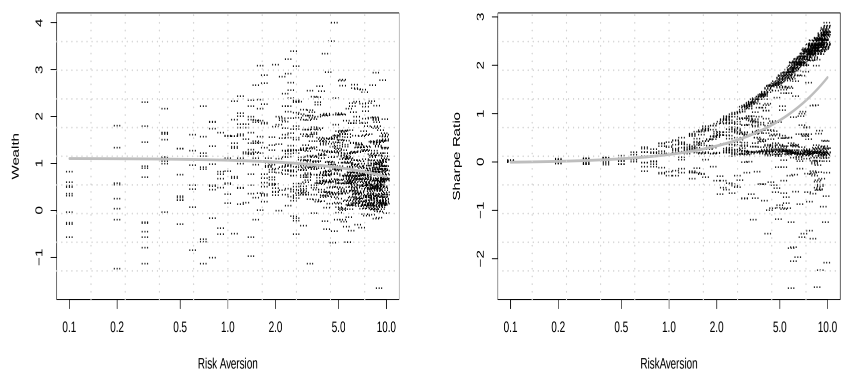
We continue to increase the number of traded assets. We now consider a 20-asset long-only case. The simulation results are presented in figures 2(a) and 2(b). Wealth has not such a sharp increase as in the 3-asset case (the linear regression coefficient now equals 0.02977): it rather increases smoothly with the risk aversion increasing. This behavior can be explained by the fact that the portfolio composition is affected differently by the changes in A for different number of assets classes that constitute the optimal portfolio [11]. The Sharpe ratio has an increasing dynamics when risk aversion increases, but the difference between the maximum and the minimum values is relatively small ($1.644210 - 1.502465 = 0.141745$). Thus, we can conclude that risk aversion has a relatively small effect on the variations of the Sharpe ratio.



(a) Wealth Distribution for Agents with Different Risk Aversions. *Wealth Increasing* $= \frac{W_t}{W_0}$. *Regression coefficient*$= 0.02977$

(b) Sharpe Ratio Distribution for Agents with Different Risk Aversions. *Regression coefficient*$= 0.01244$

Fig. 2 20-asset long-only case. Each point is the averaged value of 100 simulations. X axis is in log-scale



(a) Wealth dynamic for agents with different risk aversions. *Wealth Increasing* $= \frac{W_t}{W_0}$. *Regression coefficient*$= -0.03942$

(b) Sharpe Ratio dynamics for agents with different risk aversions. *Regression coefficient*$= 0.17709$

Fig. 3 20-asset short-selling case. Each point is the average value of 100 simulations. X axis is in log-scale

For the short-selling case, a risk-free asset is introduced. As soon as short-selling is allowed, one part of the riskier agents runs out of the market, while other agents with the same risk preferences $A < 1$ obtain a much higher wealth than in the constrained-portfolio case. Thus, there are two possibilities for the riskier agents : either to lose their initial endowment, or to increase their wealth by a factor much higher than the one of risk-averters. The conservative agents(risk-averters) on the one hand have a moderate wealth increase factor, on the other hand, they have less chances to lose their initial wealth (see figure 3(a)).

Figure 3(b) as well as the regression coefficient (0.17709) show that, contrary to the constrained portfolio situation, risk aversion has a significant effect on the Sharpe Ratio when short selling is allowed. Even if the Sharpe ratio distribution exhibits a higher variance when risk aversion increases, conservative agents tend to considerably improve their Sharpe Ratio. We can conclude that in the unconstrained portfolio framework it is better to be risk averse and to invest in risk-free assets.

3.1 Ecological Competition Analysis of Strategy Performance

Next, we compare the relative performance of investment strategies using Ecological Competition [18], [26], where agents change strategies between the trading periods using their historical performances. This research approach is widely used to understand nonlinear dynamical systems in which two or more species or agents interact through competition for resources. Stock market can be considered as the environment with the agents competing for the value of traded stocks. Traders run out of the market and change strategies that performed well during the last round. The agents populations compete each against the others in order to get higher wealth or Sharpe ratio. This approach not only allows us to track a particular performance measure, but also to follow its evolution in the long-run. Additionally, ecological competitions show the effects of each strategy on the others. For instance, one population of agents can take advantage from the presence of the others.

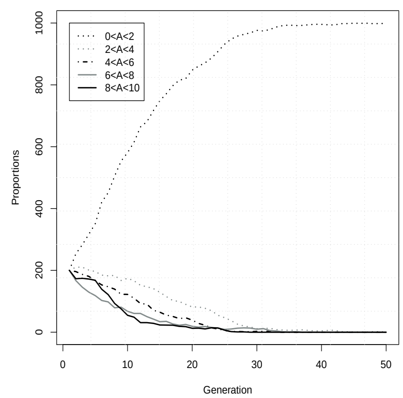
Initially, we consider an environment with $N = 10$ types of traders with populations x_i $i = \overrightarrow{1, N}$ in equal proportions (100 agents each) who interact through trading in order to obtain the highest possible wealth. The only difference in population strategies is the risk preferences. $(i-1) < A_i \leq i$ is the risk aversion measure for population x_i, $i = \overrightarrow{1, 10}$. The total number of agents is $X = \sum_{i=1}^{N} x_i = 1000$ and remains constant over the simulations. The proportions of the populations are updated every simulation round according to their gained wealth $x_i = X\frac{W_i}{W_T}$, where W_i is the wealth gained by agents population i, W_T is the total wealth. The agents population are said to be run out-of the market if $x_i = X\frac{W_i}{W_T} < 1$. The same partition principle is used with the Sharpe ratio instead of the wealth criterion. We study separately two cases of portfolio construction, the unconstrained one, (short selling allowed) and the constrained (long-only) portfolio with 20 assets.

Ecological Competition Analysis: Short Selling Allowed

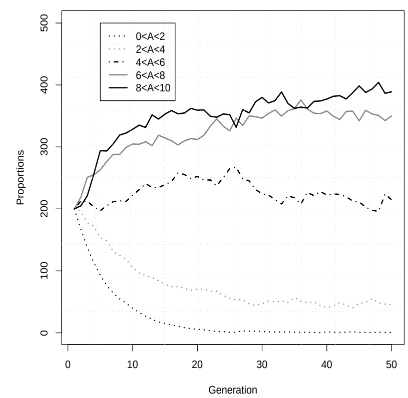
Figures 4(b) and 4(a) confirm the results highlighted only with the end-of-period results (see figures 3(a) and 3(b)). When short selling is allowed, the risk lovers compete others in terms of wealth but quickly run out-of the market in the competitions for where the Sharpe Ratio is used as a performance measure.

Ecological Competition Analysis: Long-only

In the case of long-only constrained portfolio, the figure 5(a) shows the highest (as well as the lowest) risk aversion values do not guarantee the highest earnings. Risk lovers $(0 < A \leq 2)$ as well as absolute risk averters $(8 < A \leq 10)$ run quickly out-of the competitions (in ≈ 100 rounds). Only the traders with a moderate level of risk aversion $4 < A < 6$ survive in the long run (> 500 rounds).



(a) Wealth dynamics for agents with different risk aversions.



(b) Sharpe ratio dynamics for agents with different risk aversions.

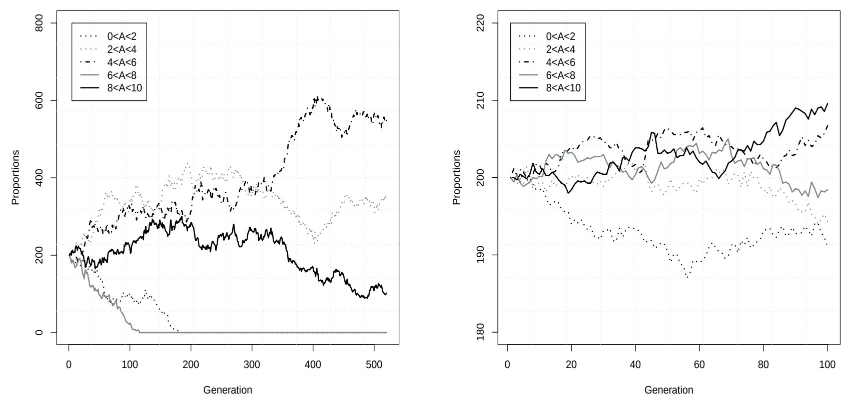
Fig. 4 Ecological competitions: 20 assets, short-selling allowed. Strategies are grouped by two for the sake of results tractability

In the competition based on risk adjusted returns, the conservative traders slightly outperform the aggressive ones (figure 5(b)). These results are consistent with those previously presented in figure 2(b). A similar conclusion emerges: risk aversion does not have a significant impact on the Sharpe ratio improvement.

If the market is populated by agents with constrained portfolios and agents with unconstrained portfolios, the traders using short-selling easily win the competition for wealth. A possible explanation for this phenomena is that the portfolio performance is improved because traders sell the assets that outperform ("sell overprices assets") and buys the assets that underperform during the trading period ("buy underpriced assets"). According to [17] the long-only strategies have zero-positions ($\alpha_{j,t}^{i,*} = 0$) in about 50% of the traded assets. Thus, the agents with long-only strategies rebalance only half of their investment set to maintain their target weights. At the same time, the agents with short-selling strategies trade the whole set of assets, and increase their wealth more efficiently.

4 Conclusion

By overcoming models based on fixed proportions of agents, we conclude that the final wealth as well as agents' risk adjusted return not only depend on



(a) Wealth dynamics for agents with different risk aversions.

(b) Sharpe ratio dynamics for agents with different risk aversions.

Fig. 5 Ecological competitions: 20 assets, long only. Strategies are grouped by two for the sake of results' tractability

their accuracy to predict expected returns and covariances of assets, but also on their risk preferences.

Our model based on ecological competition characterizes the evolution of agent populations when traders switch from the old strategy to a new one (by adjusting their risk preferences) according to its performance in the past.The main assumption is that all agents belonging to a group share the same risk preferences (risk aversion range $[A_{min}, A_{max}]$), but are allowed to change groups between the trading periods. In such a way, the fraction of agents using the same strategy characterizes its success in the past.

Our extensive simulations demonstrate that when short selling is allowed, the risk lovers ($A < 2$)compete others on the wealth basis on one hand but, on the other hand, quickly run out-of the market in the competitions based on the Sharpe Ratio. Only conservative traders survive in the long run.
However, when short selling is forbidden (long-only case), the highest, as well as the lowest risk aversion do not guarantee the highest earnings. Aggressive ($A < 2$) and strongly conservative ($A > 8$) traders drive quickly out-of the competition for wealth. Conservative traders beat aggressive traders in the competition for higher Sharpe ratio of portfolio. Finally, the current work shows the importance of degree of risk aversion in agents' survivability in a long run.

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