

## Nash Welfare Allocation Problems: Concrete Issues

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**Abstract**—The allocation of  $m$  resources between  $n$  agents is an AI problem with a great practical interest for automated trading. The general question is how to configure the behavior of bargaining agents to induce a socially optimal allocation. The literature contains many proposals for calculating a social welfare but the Nash welfare seems to be the one which has the most interesting properties for a fair agent society. It guarantees that all resources are fairly distributed among agents respecting their own preferences. This article shows first that the computation of this welfare is a difficult problem, contrary to common intuition. Many counter-examples describe the pitfalls of this resolution. In a second step, we describe our distributed multi-agent solution based on a specific agent's behavior and the results we get on difficult instances. We finally claim that this anytime solution is the only one able to effectively address this problem of obvious practical interest.

**Keywords**-Resource allocation; Nash welfare; multi-agent system; negotiations

### I. INTRODUCTION

Allocation problems can be encountered everywhere in real life through countless applications. Their aims is the determination of an allocation maximizing (or minimizing) a given objective function. However, this seeming simplicity hides very rich issues. The optimal resolution of allocation problems is most of the time very complex, mainly due to unscalability. A lot of studies have been performed on distributed approaches [2], [3] and centralized ones [9]. In these studies, different assumptions are made and various welfare functions are studied.

In this article, we are interested in the Nash welfare [1], which from our point of view has interesting properties. For instance, no agent is neglected when maximizing the Nash welfare (as soon as  $m > n$ ), and it also avoids the draining of resources by an agent with low preferences. Optimal Nash allocations have then nice properties, which are not always satisfied by allocations maximizing other welfare notions (like the utilitarian or the egalitarian welfare, which are more widely used up-to-now). Furthermore, the Nash welfare is independent of utility scales and normalizes agents' utilities.

In spite of its qualities, this notion is barely studied up to now. We show in this article that there does not exist incremental methods to identify Nash optimal allocations. Consequently, an explicit enumeration of all possible allocations is required to achieve a best Nash allocation. However,

an allocation problem based on  $n$  agents and  $m$  resources leads to  $n^m$  distinct allocations. The explicit enumeration is not scalable<sup>1</sup> and then we have to design an alternative method to find a global optimum for the Nash welfare.

This paper presents two main contributions. First, Section II enumerates the difficulties and wrong ideas about allocation problems, either in distributed settings, or in centralized ones. Then, Section III describes an efficient solution to Nash allocation problems in a distributed way.

### II. USUAL WRONG IDEAS ON NASH ALLOCATION PROBLEMS

In order to show the different issues related to Nash allocation problems, we have to introduce few notations used in this paper. The first important point is to stress that in all this article, we consider only the Nash welfare. The allocation problem is defined on a population  $\mathcal{P}$  of  $n$  agents and on a set  $\mathcal{R}$  of  $m$  resources. Each agent  $a_i$  owns a bundle denoted by  $\mathcal{R}_{a_i}$  containing its resources.

Moreover, we assume that each agent expresses preferences over the resource set, and we suppose that these preferences are given using a normalized utility function. This utility function is an additive function  $u_{a_i} : 2^{\mathcal{R}} \rightarrow \mathbb{R}$ :

**Definition 1** (Utility function). When agent  $a_i \in \mathcal{P}$  owns a set of resources  $\mathcal{R}_{a_i} \subseteq \mathcal{R}$ , its utility is evaluated as follows:

$$u_{a_i}(\mathcal{R}_{a_i}) = \sum_{r \in \mathcal{R}_{a_i}} u_{a_i}(r), \quad a_i \in \mathcal{P}, \mathcal{R}_{a_i} \subseteq \mathcal{R}.$$

Let us note  $\mathcal{A}$  the set of all possible allocations. A deal  $\delta$  changes an allocation  $A$  into a new allocation  $A'$ :  $\delta(A) = A'$ .  $\mathcal{T}$  is the set of deal kinds allowed between agents (gifts, swaps, ...).

#### A. The Nash welfare, an interesting notion

The Nash welfare is an interesting notion from the social choice theory [1], [5]. This notion is barely used in practice in spite of interesting properties. Let us first define this social welfare:

<sup>1</sup>From the birth of the solar system ( $4.6 \cdot 10^9$  years) with a computer determining 1 million allocations per second, we would have solve a Nash problem with 10 agents and 23 resources.

**Definition 2** (Nash welfare). The Nash welfare of an allocation  $A \in \mathcal{A}$ , denoted by  $sw_n(A)$ , corresponds to the product of the individual welfare of all agents of the population  $\mathcal{P}$ :

$$sw_n(A) = \prod_{a_i \in \mathcal{P}} u_{a_i}(\mathcal{R}_{a_i}), \quad A \in \mathcal{A}.$$

Maximizing the Nash welfare can be considered as a compromise between maximizing the average richness and minimizing inequalities in the population. A question can then be raised: why such an interesting notion is seldom used in practice?

### B. Two classes of allocation problem

The first preconceived ideas is independent of the welfare notion considered, and is related to the kind of optimum required. All allocation problems do not have the same purpose, and are not based on similar assumptions. On one side, the aim is to identify an optimal allocation. In such cases, resources are considered separately from agents, and centralized techniques can be used efficiently. The *winner determination problem* in combinatorial auction belongs to this class of problems [9]. On the other side, the aim is to determine a path of deals leading from an initial allocation to an optimum. To find such a path, distributed algorithms based on deals between agents are often used [2], [3], [6]. Indeed, centralized algorithms are unadapted, mainly due to scalability issues, as very higher number of possible resource allocations exists. However, optima from both cases may be different: a path of deals  $\delta \in \mathcal{T}$  from an initial allocation to a global optimum may not exist.

**Example 1.** Let us design an example based on a population  $\mathcal{P} = \{a_1, a_2\}$  and  $\mathcal{R} = \{r_1, r_2, r_3\}$ . Agents' preferences are described in the next table, which contains the utility value associated with each resource by each agent.

$u_{a_i}(r_j)$		$\mathcal{R}$		
		$r_1$	$r_2$	$r_3$
$\mathcal{P}$	$a_1$	5	5	1
	$a_2$	1	1	5

Let us now assume that the initial allocation is the following:  $A = [\{r_1\}\{r_2, r_3\}]$  where agent  $a_1$  owns  $r_1$  while agent  $a_2$  owns other resources  $r_2$  and  $r_3$ . The welfare value associated with it is  $sw_n(A) = 30$ . It is quite easy to determine the optimal allocation  $A' = [\{r_1, r_2\}\{r_3\}]$ , which is associated with  $sw_n(A') = 50$ .

However, no path of swap deals (the exchange of one resource against one other resource) can reach the optimum Nash allocation, while a path of gift deals (the gift of one resource without counterpart) leading to an optimum exists. Indeed, the exclusive use of swaps prevents changes in the number of resources per agent.

Both kinds of optima should be distinguished since they correspond to optimal solutions provided by different approaches. If the welfare values of both optima are similar, it

means that a social optimum can be achieved in a distributed way, depending on the negotiation settings.

**Definition 3** (Global optimum). A resource allocation  $A \in \mathcal{A}$  is a global optimum if no other resource allocation  $A' \in \mathcal{A}$  associated with a greater social value exists.

$$\nexists A' \in \mathcal{A} \quad sw_n(A') > sw_n(A) \quad A, A' \in \mathcal{A}, A \neq A'.$$

**Definition 4** ( $\mathcal{T}$ -optimum). A resource allocation  $A \in \mathcal{A}$  is a  $\mathcal{T}$ -optimum if no path of deals, belonging to the set of allowed deals  $\mathcal{T}$ , leads to a resource allocation associated with a greater social welfare value.

$$\forall A' \in \mathcal{A}, \quad \nexists \delta \quad sw_n(A') > sw_n(A) \quad \delta \in \mathcal{T}, A \in \mathcal{A}.$$

A  $\mathcal{T}$ -optimum cannot be greater than a global optimum, whereas the inverse is usually true. In Example 1, allocations  $A$  is a *swap-optimum* whereas it is not a global optimum. The determination of a  $\mathcal{T}$ -optimum cannot be handled using centralized algorithms, due to scalability issues.

### C. Limits of linear programming

However, even if we are “only” interested in an globally optimal allocation, centralized techniques is inefficient for the Nash welfare. The centralized solving of Nash allocation problems can be formulated by means of a mathematical model using variables  $x_{ar}$  describing the ownership of a resource  $r \in \mathcal{R}$  by an agent  $a \in \mathcal{P}$ :

$$x_{ar} = \begin{cases} 1 & \text{if agent } a \text{ owns resource } r \\ 0 & \text{otherwise} \end{cases} \quad r \in \mathcal{R}, a \in \mathcal{P}.$$

Then, the Nash allocation problem can be formulated as follows:

$$sw_n^* = \begin{cases} \max \prod_{a \in \mathcal{P}} \sum_{r \in \mathcal{R}} u_a(r) x_{ar} \\ \text{s.t: } \sum_{a \in \mathcal{P}} x_{ar} = 1 & r \in \mathcal{R} \\ x_{ar} \in \{0, 1\} & r \in \mathcal{R}, a \in \mathcal{P}. \end{cases}$$

From the expression of the Nash welfare function, we can see that this welfare function has no nice mathematical property. Indeed, this function is not concave, not convex, and not linear. Solving quadratic problems (i.e. limited to 2 agents) is a complex task and classic optimization techniques are not efficient as soon as  $n > 2$  agent are considered. In order to transform the optimization of a product into the optimization of a sum, the use of the logarithm function represents an intuitive solution.

$$\log(a * b) = \log(a) + \log(b).$$

However, this transformation does not solve the problem at all! Indeed, the logarithm is not a linear function and it is not easier to optimize a sum of non-linear functions. Optimization software like CPLEX or MATLAB are inefficient. Hence, this transformation change a complex function into another complex function, which is not more convenient.

Since the optimization way is not efficient in this case, we can look for an incremental approach, based on the enlargement from an optimal solution of a sub-part of a whole problem.

#### D. Impossible incremental methods

The design of an incremental method may represent an interesting approach. The idea is to use the optimal solution of a restricted part of the initial problem as a basis to determine an optimal solution of the complete problem. A sub-problem can be obtained considering either a subset of resources, or a subset of agents, or both. In this section, a sub-problem considers a subset of the resources  $\mathcal{R}' \subset \mathcal{R}$  and aims to determine an optimal Nash allocation on  $\mathcal{R}'$ . New elements  $\{\rho \subseteq \mathcal{R} \setminus \mathcal{R}'\}$  are added to the restricted problem, and we try to adapt the optimal solution of the restricted problem to determine an optimal solution of the new problem. A method could be to allocate first one resource to all agents using the *Hungarian* algorithm, and then to allocate the remaining resource.

**Proposition 1** (No incremental method). *It is not possible to determine a global optimum of the complete problem based on a globally optimal solution of a restricted part of the problem.*

*Proof:* In order to illustrate this proposition, let us consider a counter-example based on  $\mathcal{P} = \{a_1, a_2\}$  and  $\mathcal{R} = \{r_1, r_2, r_3\}$ . Agents' preferences are described as follows:

$u_{a_i}(r_j)$		$\mathcal{R}$		
		$r_1$	$r_2$	$r_3$
$\mathcal{P}$	$a_1$	4	3	4
	$a_2$	2	4	7

Let us consider the sub-problem of optimally allocating  $\mathcal{R}' = \{r_1, r_2\}$  to agents of  $\mathcal{P}$ . In this case, only four allocations are possible. The one maximizing the Nash welfare is:  $A = [\{r_1\}\{r_2\}]$  associated with the value  $sw_n(A) = 4 \times 4 = 16$ .

Let us now determine the global optimum for the complete problem. Eight allocations are now possible, and the global optimum is:  $A' = [\{r_1 r_2\}\{r_3\}]$  associated with the Nash value  $sw_n(A') = 7 \times 7 = 49$ .

In order to obtain the global optimum  $A'$ , it is mandatory to release the resources previously allocated in  $A$ . Hence, it is not possible to determine a global optimum using an incremental technique, constructing solutions little by little. ■

We can also consider another kind of sub-problem defined with  $n_1 < n$  agents and  $m$  resources. However, it is obvious that even if we distribute optimally all  $m$  resources between 2 agents, it is mandatory to redistribute all of them if a new agent joins the population (otherwise the new agent would get a utility of 0). Hence the optimal solution to a

sub-problem does not help to solve the whole problem. This proposition explains why neither *branch& bound* algorithms nor *constraint programming* are efficient. It has important consequences on the design of solving algorithms.

Note that the proposition is still valid where the egalitarian welfare is considered, but not when the utilitarian or the elitist welfare notions are considered.

#### E. How reliable are heuristics?

Since the determination of the Nash optimum is difficult, either by centralized techniques or incremental methods, the design of heuristics may represent a solution to estimate the Nash value of the global optimum.

We used various techniques to design a lot of heuristics. For instance, we tested the sequential allocation to the agents of the most valuable available resources, according to their preferences, the allocation of the resources to one of the agent which evaluate it the most, some of them are based on the Hungarian method, ...

We implement many of them and compare their results in a tournament but these results are not detailed here due to space restrictions. The heuristic that achieves best results most of the time is composed by two steps. The first one is to allocate each resource to the agent which values it the most. The second step is to check that all agents have at least one resource, otherwise it looks for an agent that can give one of its resources, maximizing locally the product of both agent utility.

Since these algorithms are heuristics, by definition, they only estimate the value of the globally optimal solution. It is legitimate to question the quality of the solution provided by these heuristics. A relative comparison between values provided by each heuristic is not sufficient to guarantee the quality of the best one. It is possible that the best heuristic only provides a value far from the global optimum. Since the only way to certify the optimality of a solution is the explicit enumeration, only very small instances can be solved. In other cases, with a large population and a large set of resources, it is not possible to guarantee the reliability of the provided results, i.e. that the used heuristics provides a global optimum.

In the previous section, we focus on allocation problems, where resources are not allocated initially to agents' bundle. Since these centralized techniques are not scalable, alternative methods based on agent negotiations have been developed to solve such issues.

#### F. Individual rationality and efficiency

In agent-based methods using deals, agents are often assumed autonomous. The initial allocation evolves step by step, by means of local deals between agents. Instead of having a central entity which decides how to allocate all resources between agents, the decision is distributed to the agent level. Each agent has to determine locally which

deals are profitable. This decision-making is based on an acceptability criterion.

In literature (e.g. [8]), the most widely used criterion is the *individual rationality*: this criterion focuses on the private satisfaction of an agent. It is mainly used in the case of selfish agents.

**Definition 5** (Rational agent). A rational agent only accepts a deal that increases its own utility value. If the agent  $a_i \in \mathcal{P}$  is rational, an acceptable deal must satisfy the following condition:

$$u_{a_i}(\mathcal{R}'_{a_i}) > u_{a_i}(\mathcal{R}_{a_i}), \quad a_i \in \mathcal{P}, \mathcal{R}_{a_i}, \mathcal{R}'_{a_i} \subseteq \mathcal{R}.$$

This acceptability criterion is based on personal information, so it is quite easy to compute locally the welfare value. However, considering its individual welfare is quite inefficient to achieve socially optimal allocation in a population. Indeed, only very few rational deals can be performed, and the population stays in a poor situation.

**Example 2.** In order to illustrate this property, let us design a small example. It is based on  $\mathcal{P} = \{a_1, a_2\}$  and  $\mathcal{R} = \{r_1, r_2\}$ . Agents' preferences are described as follows:

$u_{a_i}(r_j)$		$\mathcal{R}$	
		$r_1$	$r_2$
$\mathcal{P}$	$a_1$	1	4
	$a_2$	10	5

Let us assume that the initial allocation is  $A = [\{r_1\}\{r_2\}]$ , and then the initial Nash welfare is  $sw_n(A) = 5$ . From this allocation, no rational deal can be performed. No gift is rational since the agent that gives a resource will decrease its individual welfare. Similarly, the lone resource swap would decrease the individual welfare of agent  $a_1$  from 5 to 4.

However, there exist an allocation associated with a much larger welfare value. Indeed,  $A' = [\{r_2\}\{r_1\}]$  is associated with:  $sw_n(A') = 40$ . This global optimum cannot be achieved using only rational deal.

This example shows how easily resources can be trapped in an agent's bundle, and thus prevent the achievement of globally optimal allocations. As shown in [7], according to the negotiation, the efficiency of rational negotiations never exceeds 20% of the global optimum.

Note that this result on the inefficiency of the use of *individual rationality* in practice is valid independently of the social welfare notion considered.

### G. Restriction on deals

A deal is characterized by the number of agents involved and by the number of resources each of them can offer. The two main classes of deals are *bilateral* ones and *multilateral* ones.

First, bilateral deals involves only two agents at a time: the initiator and a partner. Different types of bilateral deals have

been classified in [8]. It is the most widely used deal class in literature, due mainly to scalability reasons. Multilateral deals may involve many agents simultaneously [4]. Deals of this class are barely used in practice because their identification is an issue which cannot be solved efficiently in a distributed way. The number of possible deals increases exponentially with the number of agents involved. Thus, in practice, restrictions on the number of participants are used to decrease the exponential complexity. However, imposing restrictions on deals may prevent the achievement of optimal allocations.

**Proposition 2.** *Within a population  $\mathcal{P}$  of  $n$  agents, a deal involving simultaneously  $n$  agents may be required to achieve a globally optimal allocation.*

**Corollary 1.** *Restricting the number of agents that can be involved simultaneously in a deal may prevent the achievement of globally optimal allocations.*

*Proof:* In order to prove this property, let us design a small counter example. It is based on  $\mathcal{P} = \{a_1, a_2, a_3\}$  and  $\mathcal{R} = \{r_1, r_2, r_3\}$ . Agents' preferences are described as follows:

$u_{a_i}(r_j)$		$\mathcal{R}$		
		$r_1$	$r_2$	$r_3$
$\mathcal{P}$	$a_1$	2	1	5
	$a_2$	5	2	1
	$a_3$	1	5	2

Let us assume that the initial resource allocation is:  $A = [\{r_1\}\{r_2\}\{r_3\}]$ , which is associated with  $sw_n(A) = 8$ . No sequence of acceptable deals can lead to a better allocation. Indeed, gifts are not rational, and all swaps leads to the decrease of the individual welfare of an agent. Hence, this allocation can be the one provided at the end of a negotiation process. However, this allocation is not optimal since it exists  $A' = [\{r_3\}\{r_1\}\{r_2\}]$  such that  $sw_n(A') = 125$ .

The only way to achieve the optimum corresponds to three simultaneous gifts. Indeed, if the three agents give their resource to one partner, and receive another resource from the other partner simultaneously, they all improve their individual utility. Thus, imposing restrictions on the number of agents that can be involved in a deal may prevent the achievement of optimal allocation. ■

This property is also satisfied when other welfare functions are considered, or when other acceptability criteria are considered.

### H. Deal decomposition

In order to identify an acceptable deal in a scalable way, the maximum number of resources that agents can offer is usually restricted. For instance, a restriction to one resource means that agents can only performed gifts or swaps (one resource against nothing or one resource against another one). However, a question can be legitimately raised: is these



limitations impact the efficiency of the negotiation? In other words, does the bound on the maximum number of resources offered may limit the efficiency of the solving process?

**Proposition 3** (Nash deal decomposition). *Deals cannot always be split into a sequence of acceptable deals of lesser cardinality.*

*Proof:* Let us prove this proposition using the smallest counter-example. Let us consider a population of two agents  $\mathcal{P} = \{a_1, a_2\}$  and a set of two resources  $\mathcal{R} = \{r_1, r_2\}$ . Agents' preferences are described as follows:

		$\mathcal{R}$	
		$r_1$	$r_2$
$\mathcal{P}$	$a_1$	1	7
	$a_2$	7	1

Let us assume that the initial allocation is  $A = [\{r_1\}\{r_2\}]$ , associated with  $sw_n(A) = 1$ . A swap is acceptable for both agents because they improve their individual welfare. It leads to an allocation  $A' = [\{r_2\}\{r_1\}]$ , which is associated with  $sw_n(A') = 49$ .

However, this acceptable deal  $\delta$  cannot be split into a sequence of acceptable deals. The only way to decompose this deals is a sequence of two gifts. But rational gifts cannot be acceptable since all utility values are positive. Thus, a rational deal cannot be always decompose into a sequence of acceptable deals of lesser cardinality. ■

The first consequence of this proposition is quite obvious: large acceptable deals cannot be decompose into a sequence of acceptable deals of lesser cardinality, and then into a sequence of gifts. The other consequence of this proposition is to claim that the largest deals are mandatory to guarantee the achievement of best allocations. It may be required for an agent to offer its complete resource bundle against the whole bundle of its partner.

These results have been proved in this section in the context of rational agents, but counter-examples can also be designed when other acceptability criteria are used.

### I. Nash negotiations and social graphs

All former studies never consider a facet of negotiations that occurs in most applications. One can consider that they make an unrealistic assumption. Indeed, in most agent-based negotiation studies (e.g. [2], [3]), agents communication abilities are not restricted. An agent can usually negotiate with all other agents in the population whereas it is not the case most of the time. For instance, in a peer-to-peer network, a peer does not know all other peers in the network. In a social network on the Internet, a person does not know all other members of the social network. In such applications, an agent is not even aware of the whole system, and must based its decision on local information only.

Since none of former studies consider restrictions on communication abilities, it is one more time legitimate to

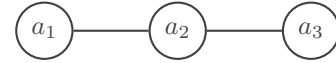
investigate the importance of such a parameter. Indeed, negotiation processes, which lead to optimal solutions according to complete communication possibilities (i.e., based on complete social graphs), may only lead to solutions far from the optimum, when communications are restricted.

**Proposition 4** (Social graph impact). *Independently of the objective function considered, a restricted social graph may prevent the achievement of optimal resource allocations.*

*Proof:* Let us prove this proposition using a counter-example, based on a population  $\mathcal{P} = \{a_1, a_2, a_3\}$  and a set of resources  $\mathcal{R} = \{r_1, r_2, r_3\}$ . The agents' preferences are described as follows:

		$\mathcal{R}$		
		$r_1$	$r_2$	$r_3$
$\mathcal{P}$	$a_1$	3	1	9
	$a_2$	1	4	1
	$a_3$	10	2	3

The social graph describing the agent communication abilities is represented next:



According to the topology of this social graph, agent  $a_2$  can communicate with agent  $a_1$  and  $a_3$ , while they can only communicate with  $a_2$  but not between them. The initial resource allocation is  $A = [\{r_1\}\{r_2\}\{r_3\}]$  associated with  $sw_n(A) = 36$ .

Only two resource swaps are possible. Agents  $a_1$  and agent  $a_2$  can exchange  $r_1$  and  $r_2$ , or agents  $a_2$  and agent  $a_3$  can exchange respectively  $r_2$  and  $r_3$ . Both cases lead to a decrease of the utility of at least one participant. Thus, no acceptable exchange is possible.

However, this allocation  $A$  is not an optimal allocation. Indeed, the swap of  $r_1$  and  $r_3$  by agents  $a_1$  and  $a_3$  would lead to a better allocation  $A' = [\{r_3\}\{r_2\}\{r_1\}]$ , associated with  $sw_n(A') = 360$ . Hence, due to the topology of the social graph, restricting the agent communication abilities, the negotiation process cannot achieve an optimal solution. ■

Considering the social graph also has an indirect influence on the negotiation processes. While it may not be important to consider the order in which agents negotiate when the social graph is complete, this order becomes essential when communications between agents are restricted. Indeed, without restriction, resources can always be traded with all other agents.

**Proposition 5** (Negotiation order). *Independently of the objective function which is considered, the order in which agents negotiate with each other may prevent the achievement of optimal resource allocations.*

*Proof:* The proposition can be proved using a counter-

example. Let us consider a population  $\mathcal{P} = \{a_1, a_2, a_3\}$  and a set of resources  $\mathcal{R} = \{r_1, r_2, r_3\}$ . The agents' preferences are described as follows:

$u_{a_i}(r_j)$	$\mathcal{R}$		
	$r_1$	$r_2$	$r_3$
$a_1$	2	10	4
$a_2$	5	3	9
$a_3$	2	7	1

The social graph describing the agent communication abilities is represented next:



According to the topology of this social graph, agent  $a_2$  can communicate with agent  $a_1$  and  $a_3$ , while they can only communicate with  $a_2$  but not between them. The initial resource allocation is  $A = [\{r_1\}\{r_2\}\{r_3\}]$  associated with  $sw_n(A) = 6$ .

Let us now assume that agent  $a_2$  initiates a negotiation. According to the topology, two partners are possible. Depending the initiator choice, the negotiation process may end with a sub-optimal allocation. Indeed, if agent  $a_2$  negotiates first with agent  $a_1$ , the allocation achieved is  $A' = [\{r_2\}\{r_1\}\{r_3\}]$  and is associated with  $sw_n(A') = 50$ . However, if  $a_3$  is chosen, the allocation achieved is  $A'' = [\{r_1\}\{r_3\}\{r_2\}]$  associated with  $sw_n(A'') = 126$ .

Hence, the order of negotiation becomes an important parameter to consider when the communication abilities are restricted. ■

In this section, we show that it is important to consider restricted communication abilities, as it occurs in many applications. Such restrictions represent more plausible assumptions but have important consequence on the negotiation efficiency. The topological characteristics may prevent the achievement of optimal solutions. Moreover, the order in which agents negotiate may also lead negotiation processes to sub-optimal allocations. In spite of their respective impact, these two parameters have not been considered up to now. The context of former studies can be considered as ideal while ours is more realistic.

### III. OUR DISTRIBUTED APPROACH

The previous section was dedicated to underline the pre-conceived ideas on Nash allocation problems: on centralized approaches, on the design of heuristics and on distributed negotiations. This section focuses on the negotiation settings we propose to solve efficiently Nash allocation problems.

#### A. Bilateral transactions

Since bilateral deals are the most scalable in practice, we choose to restrict agents to bilateral deals. They can be modeled generically using the number of resources each agent can offer.

**Definition 6** (Bilateral deals). A bilateral deal between two agents  $a_i, a_j \in \mathcal{P}$ , denoted by  $\delta_{a_i}^{a_j}$ , is initiated by agent  $a_i$  who involves a partner  $a_j$ . It is a pair  $\delta_{a_i}^{a_j}(u, v) = (\rho_{a_i}^\delta, \rho_{a_j}^\delta)$ , where the initiator  $a_i$  offers a set  $\rho_{a_i}^\delta$  of  $u$  resources ( $\rho_{a_i}^\delta \subseteq \mathcal{R}_{a_i}$ ) and the partner  $a_j$  offers a set  $\rho_{a_j}^\delta$  of  $v$  resources ( $\rho_{a_j}^\delta \subseteq \mathcal{R}_{a_j}$ ).

#### B. The sociability criterion

As described before, the acceptability criterion is mandatory to design a finite negotiation process. It is the basis for an agent to distinguish profitable deals from others. The most widely used criterion, namely the *individual rationality* is not efficient and leads to solutions far from the optimum. We propose a new criterion, which is more flexible and should lead to socially more interesting allocations.

**Definition 7** (Social deal). A social deal  $\delta$ , which changes the initial resource allocation  $A$  to a new one  $A'$ , is a deal leading to an improvement of the social welfare.

$$sw_n(A') > sw_n(A), \quad A, A' \in \mathcal{A}.$$

The social criterion is centered on the social welfare value, which is a global notion. Its value can only be determined thanks to the welfare of all agents. Agents should then know the resource bundle and the preferences of all agents in the population, in order to determine the value associated with the objective function. Such conditions cannot be satisfied since agents have only local information. The social value of the objective cannot then be locally computed. But, the computation of the exact value of the welfare function is not essential, to know its evolution is sufficient to determine whether or not a deal penalize the society. Such computations can be restricted to the local environment of agents. If participants  $a_i, a_j \in \mathcal{P}$  to a transaction consider the remaining population as a constant, the evolution of the social value can be determined on a local criterion.

$$\begin{aligned} \iff & sw_n(A) < sw_n(A') \\ \iff & \prod_{a_k \in \mathcal{P}} u_{a_k}(\mathcal{R}_{a_k}) < \prod_{a_k \in \mathcal{P}} u_{a_k}(\mathcal{R}'_{a_k}) \\ \iff & u_{a_i}(\mathcal{R}_{a_i})u_{a_j}(\mathcal{R}_{a_j}) < u_{a_i}(\mathcal{R}'_{a_i})u_{a_j}(\mathcal{R}'_{a_j}) \end{aligned}$$

#### C. The social graph

At the opposite of former studies, which always assume complete communication possibilities, solving methods based on multi-agent systems can handle the notions of *neighborhood* and *social graph*.

**Definition 8** (Neighborhood). The neighborhood of agent  $a_i \in \mathcal{P}$ , denoted by  $\mathcal{N}_{a_i}$ , is a subset of the population  $\mathcal{P}$  with whom it is able to communicate.

$$\mathcal{N}_{a_i} \subseteq (\mathcal{P} \setminus \{a_i\}), \quad a_i \in \mathcal{P}.$$

A graph of relationships, which we call a social graph, is a union of all agents' neighborhood. The social graph is a graph of relationships describing the communication abilities between the agents of a population. In such a graph, nodes represent agents, and an edge between two nodes means that the corresponding agents are able to communicate.

Different classes of graphs can be considered: *complete graphs* (full) where an agent can always communicate with all other agents, *structured graphs* (grid) where all agents have the same number of neighbors, and *Random graphs* (Erdős-Rényi, small world) where the topology is irregular.

#### D. The agent behavior

When an offer is proposed to an agent, it can choose between four alternatives. Indeed, it can accept the deal if it satisfy its acceptability criterion. Otherwise, the agent can either simply reject the deal, or change partner, or change its offer. Mixing these alternatives allow us to design behavior with different characteristics like rooted or frivolous, flexible or stubborn, ... The behavior leading negotiation processes to best results is a flexible and frivolous one, as described in the next algorithm:

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#### Algorithm 1: Frivolous and flexible agent behavior

---

```

Input: Initiator  $a_i$ 
 $L_{a_i}(\rho) \leftarrow \text{generate}(\mathcal{T}, \mathcal{R}_{a_i})$ ;
Sort  $L_{a_i}(\rho)$  according to  $u_{a_i}$ ;
Shuffle  $\mathcal{N}_{a_i}$ ;
// secondary priority on offers
forall  $\rho \in L_{a_i}(\rho)$  do
    // primary priority on agents
    forall  $a_j \in \mathcal{N}_{a_i}$  do
        forall  $\rho' \in L_{a_j}(\rho)$  do
            // deal creation
             $\delta \leftarrow (\rho, \rho')$ ;
            if ACCEPTABILITY TEST then
                Perform  $\delta$ ;
                End the negotiation;
            end
        end
    end
end

```

---

#### E. Efficient negotiation settings

The first parameter to evaluate is the **deal cardinality**. Bilateral deals  $\delta_{a_i}^{a_j} \langle u, v \rangle$  between two agents  $a_i, a_j \in \mathcal{P}$  are characterized by the number of resources offered by  $a_i, a_j$ , respectively  $u$  and  $v$ . Experiments are based on 50 agents and 250 resources (average on 100 simulations). Several negotiation policies are used and described using the cardinality parameters. The negotiation policy denoted by "up to  $\langle 2, 2 \rangle$ " means that agents can offer up to two resources during the same deal. We choose different topologies: two structured graphs and two random graphs. Figures 1 and

2 shows the evolution of the Nash welfare value during a negotiation process according to the deals cardinality.

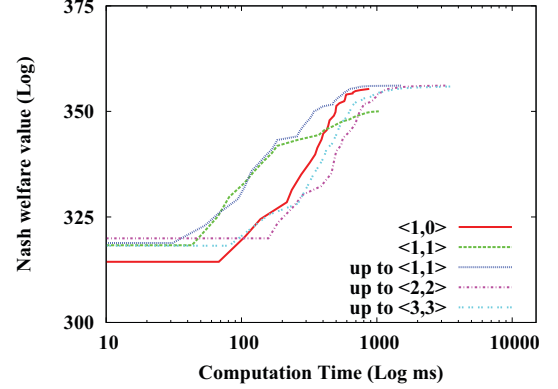


Figure 1: Negotiation efficiency vs. computation time according to the cardinality

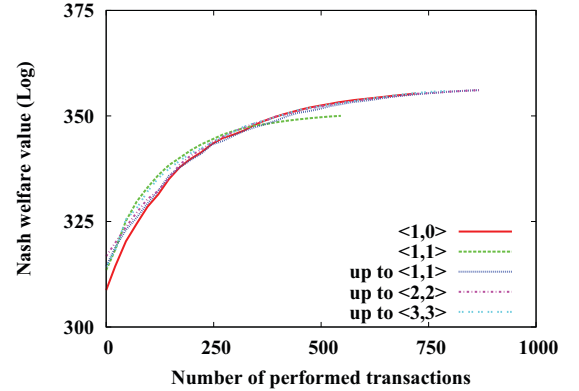


Figure 2: Negotiation efficiency vs. number of performed deal according to the cardinality

Figure 1 shows that the deals cardinality mainly affects the elapsed time. Negotiation processes based on gifts (i.e.  $\langle 1, 0 \rangle$  deals) are less time-consuming and agents perform less deals. The larger the allowed deals are, the more time consuming become negotiation processes. However, according to Figure 2 that focuses on the number of performed deals, larger bilateral deals do not improve the quality of achieved solutions. Negotiations relying only on swaps (i.e.  $\langle 1, 1 \rangle$  deals) require less deals but also achieve socially weaker allocations. All other negotiation processes end after sequences of deals of close length. Large bilateral deals do not seem to significantly improve the Nash welfare value achieved at the end of the negotiation processes. The use of deals of large cardinality does not justify the important additional costs, and thus the size of the offers should be restricted.

The next facet that should be evaluated is the **negotiation efficiency**. It can be evaluated using a comparison with the

Table I: Nash negotiation efficiency (%) according to the class of social graphs

Social graph kind	Rational criterion		Social criterion			
	$\langle 1, 1 \rangle$	up to $\langle 2, 2 \rangle$	$\langle 1, 0 \rangle$	$\langle 1, 1 \rangle$	up to $\langle 1, 1 \rangle$	up to $\langle 2, 2 \rangle$
Full	99.9	100.1	101.6	100.1	101.7	101.7
Grid	97.0	97.5	99.6	98.2	99.7	99.7
Erdős-Rényi	99.6	99.8	101.4	99.9	101.6	101.6
Small world	97.2	98.0	100.2	98.9	100.4	100.4

estimation given by centralized techniques. Table I shows the efficiency of negotiation processes based on several kinds of social graphs [6].

Table I shows that some welfare values achieved are greater than 100%. Since heuristics can only give an estimation of Nash welfare values, an efficiency greater than 100% means that negotiation processes lead to socially more interesting allocations than the ones provided by the heuristics.

Rational negotiations achieve socially weaker allocations than social negotiations. Two negotiation policies, which are based respectively on  $\mathcal{T} = \{\langle u, v \rangle | u \leq 1, v \leq 1\}$  and on  $\mathcal{T} = \{\langle u, v \rangle | u \leq 2, v \leq 2\}$ , lead to similar results. Allowing gifts and swaps during a negotiation process seems sufficient to achieve socially efficient allocations. Larger deals do not significantly improve the Nash welfare values achieved while the negotiation cost increases a lot.

Negotiations based on swap deals achieve the socially weakest allocations. Since the initial resource distribution cannot be modified, negotiations end quickly on local optima. The standard deviation related to negotiations based on  $\langle 1, 1 \rangle$  deals is also higher than for other deals.

Negotiation processes based on grids leads to the socially weakest allocations. The mean connectivity of the social graphs is an important feature deeply affecting the negotiation efficiency. Relationships among agents are too restricted to allow a suitable resource traffic, and then prevent the achievement of optimal allocations. The comparison between results achieved on Erdős-Rényi graphs and the ones achieved on small-worlds indicates that a large number of agents, leaves of the graph (who have only one neighbor), penalizes a lot the negotiation process.

Negotiations among social agents achieve more efficient allocations compared to rational negotiations usually studied in the literature. Negotiations based on  $\mathcal{T} = \{\langle 1, 0 \rangle, \langle 1, 1 \rangle\}$  can be considered as the best alternative to achieve socially interesting allocations. Deals of weaker cardinality are not sufficient whereas larger deals do not improve significantly the Nash welfare value while their use increases the negotiation cost. However, the exclusive use of bilateral deals cannot guarantee the achievement of a global optimum, but leads to socially close allocations instead.

#### IV. CONCLUSION

The Nash welfare, a notion with very interesting properties, is barely used in practice due to computational

limitations most of the time. A lot of related issues can be solved at the opposite of intuition. We present in the first part of this paper the usual wrong ideas on Nash allocation problems. We show the inefficiency of techniques based on linear programming, of incremental techniques and the lack of reliability of heuristics. The second part of this paper describes the distributed method we propose to efficiently solve Nash allocation problems, using deals between agents. We characterize the negotiation settings to use in order to identify a path of deals reaching optimal allocations. Any kind of contact network can be considered, and the distributed decision making is based on local information. Such assumptions correspond to a more realistic context than in former studies. We provide the behavior, the acceptability criterion and the kind of deals that should be used to reach  $\mathcal{T}$ -optimal allocations.

#### REFERENCES

- [1] K. Arrow, A. Sen, and K. Suzumura. *Handbook of Social Choice and Welfare*, volume 1. 2002.
- [2] Y. Chevaleyre, P. Dunne, U. Endriss, J. Lang, M. Lemaitre, N. Maudet, J. Padget, S. Phelps, J. Rodriguez-Aguilar, and P. Sousa. Issues in Multiagent Resource Allocation. *Informatica*, 30(1):3, 2006.
- [3] U. Endriss, N. Maudet, F. Sadri, and F. Toni. Negotiating Socially Optimal Allocations of Resources. *JAIR*, 25:315–348, 2006.
- [4] M. Hemaissia, A. El Fallah Seghrouchni, C. Labreuche, and J. Mattioli. A Multilateral Multi-Issue Negotiation Protocol. In *AAMAS'07*, pages 1–8, 2007.
- [5] H. Moulin. *Axioms of Cooperative Decision Making*. 1988.
- [6] A. Nongailard. *An Agent Based Approach for Distributed Resource Allocations*. PhD thesis, Université Lille 1 and Concordia University, 2009.
- [7] A. Nongailard, P. Mathieu, and B. Jaumard. A realistic approach to solve the nash welfare. In *PAAMS'2009*, volume 55, pages 374–382, 2009.
- [8] T. Sandholm. Contract Types for Satisficing Task Allocation: I Theoretical Results. In *AAAI Spring Symposium: Satisficing Models*, volume 99, pages 68–75, 1998.
- [9] T. Sandholm. Algorithm for optimal winner determination in combinatorial auctions. *Artificial Intelligence*, 135(1-2):1–54, 2002.