Optimal Portfolio Diversification?
A multi-agents ecological competition analysis

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Abstract In this research we study the relative performance of investment strategies scrutinizing their behaviour in an ecological competition where populations of artificial investors co-evolve. We test different variations around the canonical modern portfolio theory of Markowitz, strategies based on the naive diversification principles and the combination of several strategies. We show, among others, that the best possible strategy over the long run always relies on a mix of Mean-Variance sophisticated optimization and a naive diversification. We show that this result is robust when short selling is allowed in the market and whatever the performance indicator chosen to gauge the relative interest of the studied investment strategies.

1 Introduction

Agent-based modelling (ABM) is widely used to study economic systems under a complexity paradigm framework. Within this research stream, financial markets have received a lot of academic and practitioners interests these last years, notably in offering an alternative to mathematical finance and financial econometrics. Among the features that can be grasped with ABM, the co-evolving aspects of stock markets (investors making decisions that affect the system, which hitherto impact their behaviour along a feedback loop) is probably one of the most critical one.
In this research we actually renew the analysis of a *classical* question in Finance, namely, the relative performance of investment strategies scrutinizing their be-

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haviour in an ecological competition where populations of artificial investors co-evolve. This approach allows us to propose new results that can be compared to those of [4] or more recently [6] and [15] who did the same kind of research but within the traditional finance philosophy (no agents, no co-evolution, no complexity). In doing so, our agents populations compete each against the others with one strategy (some are based on different variations around the canonical modern portfolio theory of Markowitz [11], others on the Naive diversification principle [3] and others combine several strategies). To understand the added value of our approach compared to those of [6] and [15] one can start in summarizing their contribution before exposing the limitations of their approach.

– De Miguel and al. [6] compare several investment strategies using a *backtesting methodology*. It consists in managing a virtual portfolio of assets as if the historical prices used to run the experimentation were known. These investment strategies belong to one of the following investment rules families: i) Naive diversification or ii) Mean-Variance Diversification ("à la" Markowitz). The authors show that complicated portfolio optimization strategies not only under-perform the naive diversification, but also generate negative risk-adjusted rates of return.

– Tu and Zhou [15], extending the backtesting methodology of De Miguel and al., suggested that a combination of the 1/N strategy with the sophisticated diversification can each of its constituents taken separately. This result is proposed in an empirical framework which is extremely similar to the one of De Miguel and al.

To our opinion, the main problem with these researches is the unrealistic *"atomistic"* assumption that supports the backtesting methodology. Said simply, this assumption allows to gauge an investment strategy with historical data as if its true implementation would have not modified these prices. These assumptions in sharp contrast to analysis of [9], [7] who clearly show that prices may well be influenced by several parameters (investment strategies, the cognitive skills of investors or the market microstructure itself) that are neglected in the backtesting approach. We defend in this research that a convincing answer to the question *"among this set of investment strategies, which one outperforms the others?"*, overcoming the previously mentioned limitations, can be delivered by a multi-agent system allowing to implement ecological competitions among these strategies. We show, among others, that the best possible strategy over the long run always relies on a mix of Mean-Variance sophisticated optimization and a Naive diversification. This result reinforces the practical interests of the Markowitz framework that is strongly discussed in [6] for example.

2 Agents Behaviour

One of the advantages of ABM is that the agents are autonomous. In a mathematical model, all market participants are defined as equal-power rational entities facing homogeneous constraints. Agents actions are predetermined by strict equations describing their reaction in response to particular market conditions. ABM allows to overcome the limitations linked to that homogeneity. In this research, we design 8 agents populations, each of them following a generic strategy. These strategies are presented in subsection 2.1.

2.1 Height populations of agents based on height different generic strategies

We start by introducing how a portfolio of assets is modelled and what kind of decision agents must make in a simulation. Note again that the purpose of each strategy is to allow agents to manage a diversified portfolio of financial assets over time.

- A portfolio is defined as a vector of weights over the investment universe. This vector is denoted α^{xx}, xx allowing to identify the generic strategy determining this vector.
- Depending upon the strategy definition or the empirical design, these weights can be negative or not. If this is the case, one will refer to this situation as ”shorting allowed”, which means that agents are allowed to sell assets they do not hold provided they will repurchase them later on.
- Each time a new portfolio is computed, the current weight vector α_t^{xx} is compared to the previous one α_{t-1}^{xx} to adjust the number of stocks to hold. This adjustment take into account the weight vectors and the corresponding assets current prices. As a result agents decide to buy or to sell certain assets they hold to reach their new (weight vector) target.
- Last but not least, these decisions must be practically implemented, that means *”translated into buy or sell orders”*, with quantities and prices in accordance to the target. One must remember that each strategy implies different parameters that may have different values within the same agents population; thus each agent has his own weight vector rolling during a simulation.

This process being the same whatever the behaviour, we can now describe at fine grain the 8 generic strategies (see Table 1).

Table 1 Strategies description

Name	Short Name	Basic definition & particularities
Naive	*N*	Equal weights, no sophisticated behaviour
Mean Variance 1	*MLong*	Markowitz optimization, long positions only
Mean Variance 2	*MShort*	Same as *MLong*, shorting allowed
Market Portfolio Holders	*MP*	Weights according to assets capitalisation on the market, no sophisticated behaviour
Bayesian Traders 1	*BLong*	Based on Markowitz, estimation of moments co-moments of asset returns improved, long positions only
Bayesian Traders 2	*BShort*	Same as *BLong*, shorting allowed
Strategy Combinators 1	*CLong*	Mix of *N* and *MLong*
Strategy Combinators 2	*CShort*	Mix of *N* and *MShort*

Population 1: Naive diversification investors

The agents endowed with the naive strategy (N) ignore all information about risk and return of assets. Naive investors allocate their funds equally among the N risky assets in equal proportions $\alpha_{j,t}^{i,N} = \frac{1}{N}\ \forall j = \overrightarrow{1,N}$ the weights of wealth allocated to stock j of agent i at the moment of time t. In contrasts to sophisticated rules that are usually asymptotically unbiased but have a large (variance) estimation error in small samples, the 1/N rule is biased, but has zero estimation error.

Populations 2 and 3: Mean-variance optimizers

Agents endowed with this strategy try to minimize risk for a given target return following the mean (μ) variance (σ^2)optimisation rules introduced by Markowitz [11]. An important parameter in this process is the correlation matrix V of asset returns and the investor's risk preferences (risk aversion) defined in his quadratic utility function:

$$min\frac{1}{2}\sigma_p^2 = min_\alpha \alpha' V \alpha \tag{1}$$

$$\mu_p = \alpha' \mu \tag{2}$$

$$\sum_{i=1}^{n} \alpha_i = 1,\ \alpha^M = (\alpha_1, \alpha_2, \ldots, \alpha_n) \tag{3}$$

where n – number of assets, μ_p – expected return of portfolio, σ_p – standard deviation of portfolio, α^M – target weights defined according to Markowitz rules. This optimisation problem provides the solutions outside the range $[0,1]$, that allows *shorting*.

From its definition, we create two agents population, one allowed to use short selling (MShort), the other not allowed to do so (long only, MLong)

Population 4: Market portfolio holders

Market portfolio (MP) holder is the type of agents with a portfolio consisting of all assets in the market with weights proportional to assets capitalisation [14]. In more realistic context, if an investor has no special insights about expectation returns and volatility of individual stocks he is supposed to hold the market portfolio (portfolio of all available stocks).

$$\alpha_{j,t}^{i,MP} = \frac{P_{j,t} \times Q_{j,t}}{C_t} \tag{4}$$

$P_{j,t}$ price of asset j at moment t, $Q_{j,t}$ number of asset j traded on the market at the moment t, C_t total market capitalisation.

Population 5 and 6: Bayesian traders

Agents within this population have a behaviour that extends the Markowitz rules described in 2.1. The Markowitz approach has been criticized due to measurement errors in the estimation of assets' moments and co-moments. To overcome these problems authors like [1], [8] or [5] propose to improve the co-moments estimation in using a factor equal to $1+\frac{1}{M}$ that reduces its estimation error and leads to more reliable investment weights. Moments and co-moments being estimated following this rule, agents use equations (1)–(3) to determine the target weights.

From this logic we define two different population, one in which short selling is allowed, Bayesian Short Selling *(BShort)* and one in which it is forbidden, Bayesian Long Only strategies *(BLong)*.

Population 7 and 8: Strategy combinators

A last population has the ability at combining the naive 1/N strategy with the sophisticated Mean-Variance optimization strategy. It has been studied by some authors who thought it could improve the overall performance of investors [4]. Mathematically the weights definition of strategies combination can be describe as follow:

$$\alpha_{j,t}^{\hat{i},C} = (1-\delta)\alpha_{j,t}^{i,N} + \delta\alpha_{j,t}^{i,M} \quad \delta = \frac{\varphi_1}{\varphi_1+\varphi_2}$$
$$\varphi_2 = \frac{1}{A^2}\left[\frac{(T-2)(T-n-2)}{(T-n-1)(T-n-4)}\right] \tag{5}$$

where $\alpha_{j,t}^{i,C}$ – weights defined by strategies combination, $\alpha_{j,t}^{i,N}$ – weights defined according to naive diversification rule, $\alpha_{j,t}^{i,M}$ –weights defined according to Markowitz rule, δ – combination parameter $0 \leq \delta \leq 1$, n – number of assets, T – memory span or the length of available historical data. "Markowitz Shorting allowed" and "Markowitz Long-only" are used for combinations, hence Combination Short *(CShort)* and Combination Long *(CLong)* populations are studied in this research.

3 Simulations and Results

As mentioned earlier in this research, we have chosen to compare the relative performance of each investment strategy using an ecological simulation. We use an empirical design that is closely related to the one developed in [2]. The basic ideas governing this approach can be summarized as follows :

- Each strategy is encoded in an initial population of NNN agents. These populations are mixed and compete in the same market, trading the same stocks. Prices are the direct result of the flow of orders sent by the agents to the central order books ruling the artificial stock exchange.
- A time step in our ecological competitions is made of several *rounds*, each of them encompassing 1000 trading days.
- For such complex experiments we use the powerful Artificial Trading Open Market *(ATOM)* [12]. ATOM allows to implement thousands of agents evolving simultaneously, with different strategies.
- Initially, we populate the ATOM environment with our 8 populations of agents. The size of each of these populations x_i for $i = \overrightarrow{1,8}$ is the same $\forall i$. The total number of agents is $X = \sum_{i=1}^{8} x_i$. Populations are updated every simulation round according to their performance $x_i = X\frac{P_i}{P_T}$, where P_i the performance of population i and P_T the overall performance of the whole soup of populations. The performance can be measured as i) the total wealth (cash + market capitalization of the stocks of all the agents in each population) or ii) the average Sharpe ratio [13] of the population, during the previous round. A population is said to be extinct if $x_i = X\frac{P_i}{P_T} < 1$

Model parametrisation

The results presented later in this paper were obtained for the following parametrization of our model :

- The investment set is made of 30 different stocks (like the 40 different families of stocks listed in the CAC40 index)
- We study the 8 populations of agents presented in table 1.
- In each population, we start with 100 agents.
- 1000 days of trading form a *simulation round* in the experiments reported below. Between rounds, both stocks and cash endowments are carried forward.
- MLong, MShort, BLong, BShort, CLong, CShort use a rolling time window T=500 to estimate the necessary moments and comoments implied in the optimization process.
- All information concerning the underlying probability distribution of security prices as well as current security prices are available continuously and at no cost for all investors.
- Contrary to [6] and [15] who consider risk aversion as 1 or 3 for Markowitz strategies and its extensions, risk aversion in our simulations is uniformly drawn between [0.5,5] $A \sim D(x|0.5,5)$ in order to test a larger variety of behaviours, from risk averse agents to risk takers.
- Agents enter the market with 50 units of each type of stocks and 1000$ cash
- All agents have the same daily trading frequency, or said differently, are equally active in the market

4 Results and Discussions

We present here the results of two different ecological competitions. In the first one, the reproduction rate of each population is linked to dollars earnings (see subsection 4.1) while in the second one, it is a function of the Sharpe ratio (see subsection 4.2).

4.1 Ecological competition 1: wealth

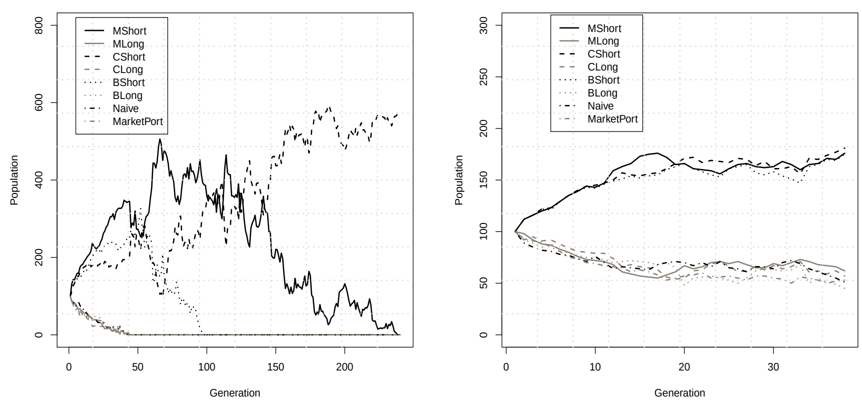
The simulations results (figure 1(a)) show that all the constrained (long-only) strategies ($MLong, BLong, CLong$), the naive (N) and the market portfolio strategies (MP) quickly disappear from the market at the end of 50 rounds. According to [10], a possible explanation of this phenomenon could be linked to the large positions (positive or negative) implied by short selling, when it is allowed : the long-only strategies have zero-positions ($\alpha^{i,*}_{j,t} = 0$) in about 50% of the traded assets. Thus, the agents with long-only strategies trade only the half of the investment set to maintain their target weights. At the same time, the agents with short-selling strategies trade the whole set of assets and increase their wealth more efficiently.

In addition, we observed that the population *CShort* are better than their individual component rules (*MShort* and *N*) which is clearly in line with the results of [15].

We also investigated a possible effect of the initial size of the population in its survival time. We therefore changed population initial distributions dynamically ($\sim D(x|20,200)$) so to get a majority of certain types of agent in the whole population soup at the beginning of each experiment. Our results indicate that even if the initial proportion of naive agents (≈ 200 individuals) is much bigger than the proportion of others (100 individuals), they cannot survive much longer in the ecological competitions where wealth rules the reproduction rate.

4.2 Ecological competition 2: Sharpe ratio

We measure the Sharpe ratio in order to estimate the agents ability to hedge the portfolio risk with many assets. Figure 1(b) reports the average evolution of agents proportions based on this indicator. As it can be observed in that figure, here again the unconstrained strategies outperform the constrained ones. These results confirm those of [10], who stress the importance of short selling in markets with many assets. At the same time, our results are not congruent with those of [6] who report that the Sharpe ratio of sample-based mean-variance strategy is much lower than that of naive strategy. One reason that could explain this discrepancy is that these authors use diversified portfolios with low volatility in their numerical simulations while our simulations rely on individual assets with more volatility.



(a) Wealth (b) Sharpe ratio

Fig. 1 Ecological competition.

5 Conclusion

Contrary to research works claiming the useless of the Markowitz theory, we show that this classical rule still outperforms the naive rules in high- and low-volatility

market regimes when the traders behaviour generate such price motions. The major benefit of our method, which relies on agent-based modelling, is its flexibility comparing with approaches of [6] and [15]. Indeed, the ABM approach allows (i) any number of traders on the market (ii) combination of large variety of strategies (iii) any number of risky assets. This flexibility provides a distinct advantage over alternative approaches to the portfolio optimisation problems.
Our findings are consistent with those of [15] and [10]. The performance of unrestricted portfolio strategies outperforms the long-only and naive strategies in both ecological competitions where the Sharpe ratio or the earnings rule the reproduction rate of the populations. Thus our result show that naively diversified portfolios are sub-optimal. Our analysis also suggest that even though the ex-ante parameters estimation of moments and co-moments involves estimation errors due to the small size of sample, the combination of mean-variance sophisticated rules and naive rules can improve the performance of their individual counterparts.

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