Algorithme de optimisation polynomiale en utilisant de bases de bord

Marta Abril Bucero, Bernard Mourrain
INRIA Sophia Méditerranée
BP 93, 06902 Sophia Antipolis, France
`Marta.Abril_Bucero,Bernard.Mourrain}@inria.fr`

Computing the global minimum of a polynomial function f on a semi-algebraic set is a difficult but important problem, with many applications. A relaxation approach was proposed in [6] which approximates this problem by a sequence of finite dimensional convex optimization problems. These optimization problems can be formulated in terms of linear matrix inequalities on moment matrices associated to the set of monomials of degree $\leq t \in \mathbb{N}$ for increasing values of t. They can be solved by Semi-Definite Programming (SDP) techniques. The sequence of minima converges to the actual minimum f^* of the function under some hypotheses [6]. In some cases, the sequence even reaches the minimum f^* in a finite number of steps [8, 16, 10, 2, 4, 14]. This approach proved to be particularly fruitful in many problems [7]. In contrast with numerical methods such as gradient descent methods, which converge to a local extremum but with no guaranty for the global solution, this relaxation approach can provide certificates for the minimum value f^* in terms of sums of squares representations.

From an algorithmic and computational perspective, some issues need however to be considered.

– *How to reduce the size of the moment matrices ?* The size of the SDP problems to be solved is a bottleneck of the method. This size is related to the number of monomials of degree $\leq t$ and is increasing exponentially with the number of variables and the degree t. Many SDP solvers are based on interior point methods which provide an approximation of the optimal moment sequence within a given precision in a polynomial time [13]. Thus reducing the size of the moment matrices or the number of parameters can improve significantly the performance of these relaxation methods. We address this issue using polynomial reduction with border basis due to their numerical stability [11, 12].

- *When is the minimum reached?* A new stopping criteria is given to detect when the relaxation sequence reaches the minimum, using a flat extension criteria from [9]. We also provide a new algorithm to reconstruct a finite sum of weighted Dirac measures from a truncated sequence of moments. This reconstruction method can be used in other problems such as tensor decomposition [1] and multivariate sparse interpolation [3]
- *How to recover the minimizer ideal?* Computing the points where this minimum is reached if they exist, is critical in many applications. Determining when and how these minimizer points can be computed from the relaxation sequence is a problem that has been adressed for instance in [5, 15] using of kernel of full moment matrices

We present a new algorithm to obtain the minimum of a real polynomial function f in a semialgebraic set $G = (G^0, G^+)$ where G^0 is a set of equalities and G^+ is a set of inequalities non negatives and we suppose that the numbers of minimizer points is finite. We compare our algorithm with the full moment matrix relaxation algorithm (implemented in c++ in the same environment that our algorithm) described in [7], which is also implemented in the package Gloptipoly of Matlab developed by D. Henrion and J.B. Lasserre.

When there are equality constraints, the border basis computation reduces the size of the moment matrices, as well as the localization matrices associated to the inequalities. This speeds up the SDP computation. In the case where there are only inequalities, the size of the moments matrices is the same but the algorithm which verifies the flat extension and the algorithm which computes the minimizers are more efficient and quicker than the reconstruction algorithm used in the full moment matrix relaxation approach. The performance is not the only issue : numerical problems can also occur due to the bigger size of the moment matrices in the flat extension test and the reconstruction of minimizers.

The experiments show that when the size of the SDP problems becomes significant, most of the time is spent during `sdpa` computation and the border basis time and reconstruction time are negligible. In all the examples, the new border basis relaxation algorithm outperforms the full moment matrix relaxation method.

Bibliographie

[1] J. Brachat, P. Comon, B. Mourrain, and E. Tsigaridas. Symmetric tensor decomposition. *Linear Algebra and Applications*, 433 :1851–1872, 2010.

[2] J. Demmel, J. Nie, and V. Powers. Representations of positive polynomials on noncompact semialgebraic sets via kkt ideals. *Journal of Pure and Applied Algebra*, 209(1) :189 – 200, 2007.

[3] M. Giesbrecht, G. Labahn, and W.-S. Lee. Symbolic-numeric sparse interpolation of multivariate polynomials. *J. Symb. Comput.*, 44(8) :943–959, August 2009.

[4] H. V. Ha and T.S. Pham. Representation of positive polynomials and optimization on noncompact semialgebraic sets. *SIAM Journal on Optimization*, 20(6) :3082–3103, 2010.

[5] D. Henrion and J.B. Lasserre. *Positive Polynomials in Control*, chapter Detecting Global Optimality and Extracting Solutions in GloptiPoly., pages 293–310. Lectures Notes in Control and Information Sciences. Springer, 2005.

[6] J.B. Lasserre. Global optimization with polynomials and the problem of moments. *SIAM J. Optim.*, 11 :796–817, 2001.

[7] J.B Lasserre. *Moments, positive polynomials and their applications.* Imperial College Press, 210.

[8] M. Laurent. Semidefinite representations for finite varieties. *Math. Progr*, 109 :1–26, 2007.

[9] M. Laurent and B. Mourrain. A generalized flat extension theorem for moment matrices. *Arch. Math. (Basel)*, 93(1) :87–98, July 2009.

[10] M. Marshall. Representations of non-negative polynomials, degree bounds and applications to optimization. *Can. J. Math.*, 61(1) :205–221, 2009.

[11] B. Mourrain and P. Trébuchet. Generalized normal forms and polynomials system solving. In M. Kauers, editor, *ISSAC : Proceedings of the ACM SIGSAM International Symposium on Symbolic and Algebraic Computation*, pages 253–260, 2005.

[12] B. Mourrain and Ph. Trébuchet. Stable normal forms for polynomial system solving. *Theoretical Computer Science*, 409(2) :229–240, 2008.

[13] Y. Nesterov and A. Nemirovski. *Interior-point polynomial algorithms in convex programming.* SIAM, Philaldelphia, 1994.

[14] J. Nie. An exact jacobian SDP relaxation for polynomial optimization. *Mathematical Programming*, pages 1–31, 2011.

[15] J. Nie. Certifying convergence of Lasserre's hierarchy via flat truncation. *Mathematical Programming*, pages 1–26, 2012.

[16] J. Nie, J. Demmel, and B. Sturmfels. Minimizing polynomials via sum of squares over gradient ideal. *Math. Program.*, 106(3) :587–606, 2006.