Structures of polyzetas and the algorithms to express them on algebraic bases on words

V.C. BUI, V. G.H.E. DUCHAMP, HOANG NGOC MINH
LIPN - Paris 13 University
99 avenue Jean-Baptiste Clément, 93430 Villetaneuse
vanchien.bui@lipn.univ-paris13.fr, gheduchamp@gmail.com,
vincel.hoangngocminh@univ-lille2.fr

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For any $(s_1, \ldots, s_r) \in (\mathbb{N}^*)^r$ with $s_1 > 1$, the polyzetas (multiple zeta values) $\zeta(s_1, \ldots, s_r)$ is defined by the following sum

$$\zeta(s_1, \ldots, s_r) := \sum_{n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}} \tag{1}$$

Let $X = \{x_0, x_1\}$ and $Y = \{y_k\}_{k \geq 1}$ be two alphabets of the set of Lyndon words denoted by $\mathcal{L}ynX$ and $\mathcal{L}ynY$ respectively. Let

- $\{P_l\}_{l \in \mathcal{L}ynX}$ be a basis of the Lie algebra $\mathcal{L}ie_{\mathbb{Q}}\langle X \rangle$ and $\{S\}_{l \in \mathcal{L}ynX}$ be the (pure) transcendent basis, in duality with $\{P_l\}_{l \in \mathcal{L}ynX}$ on the Hopf algebra $(\mathbb{Q}\langle X \rangle, ., 1_{X^*}, \Delta_{⧢}, \epsilon_X, \mathcal{S})$ (see [2]),
- $\{\Pi_l\}_{l \in \mathcal{L}ynY}$ be a basis of the primitive elements of the Hopf algebra $(\mathbb{Q}\langle Y \rangle, ., 1_{Y^*}, \Delta_{\text{⧢}_-}), \epsilon_Y, \mathcal{S})$ and $\{\Sigma_l\}_{l \in \mathcal{L}ynY}$ be the (pure) transcendent basis, in duality with $\{\Pi_l\}_{l \in \mathcal{L}ynY}$ (see [1, 4, 5]).

Since, for any multi-index $(s_1, \ldots, s_r)$, the polyzeta $\zeta(s_1, \ldots, s_r)$ can be encoded by the words $x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1 \in X^*$ and $y_{s_1} \ldots y_{s_r} \in Y^*$ (see [3]) then one can define the two following non commutative generating series of polyzetas :

$$Z_{⧢} := \prod_{l \in \mathcal{L}ynX \setminus X}^{\searrow} \exp(\zeta(S_l)\, P_l) \quad \text{and} \quad Z_{\text{⧢}_-} := \prod_{w \in \mathcal{L}ynY \setminus \{y_1\}}^{\searrow} \exp(\zeta(\Sigma_l)\, \Pi_l). \tag{2}$$

Let us introduce the following non commutative generating series

$$Z_\gamma = e^{\gamma y_1} Z_{ш}. \tag{3}$$

Let Γ dedontes the Euler's Gamma function and π_Y stands for the linear projection from $\mathbb{R} \oplus \mathbb{R}\langle\langle X \rangle\rangle x_1$ to $\mathbb{R}\langle\langle Y \rangle\rangle$ mapping $x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1$ to $y_{s_1} \ldots y_{s_r}$. We will base on the following comparison formula

$$Z_\gamma = \Gamma(y_1 + 1)\pi_Y Z_{ш} \tag{4}$$

to identify the homogeneous polynomials, in weigth, among the local coordinates $\{\zeta(\Sigma_l)\}_{l \in \mathcal{L}ynY \setminus \{y_1\}}$ (and also $\{\zeta(S_l)\}_{l \in \mathcal{L}ynX \setminus X}$) upto weight 12 in Maple.

Bibliographie

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