Exploring univariate mixed polynomials

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An expression $P(z, \bar{z}) = \sum_{k=0..n} \sum_{j=0..m} a_{k,j} z^k \bar{z}^j$ where z and $\bar{z}$ are complex conjugated, is called a (univariate) mixed polynomial of bidegree (n, m). We will assume $m \leq n$ and concentrate on the case where m is small, in particular $m = 1$. Our aim is to study the roots in $\mathbf{C}$ of P. Identifying $\mathbf{C}$ with $\mathbf{R}^2$ and separating real and imaginary parts of P, i.e. writing $P = f(x, y) + ig(x, y)$ with $i^2 = -1$ and $z = x + iy$, we get a pair of real bivariate polynomials of degrees at most $n + m$. Conversely from a pair of bivariate polynomials $(f(x, y), g(x, y))$, letting $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$ and $P = f + ig$, we get a univariate mixed polynomial. However, since the two representations are different, we can investigate interesting roots structures and develop algorithms, intermediate between complex and real algebra. This representation can be also used with several variables $(z_1, .., z_l)$. It received a renewed interest with the works in Algebraic Geometry, authors investigated a new exotic sphere (à la Pham-Brieskorn), more recently Mutsuo Oka [4], thanks to mixed polynomials, answered a question of Milnor on real generalizations of Milnor fibration theorem. Roots of mixed polynomials naturally appear when expressing that a complex polynomial matrix drops rank. It also appears as Taylor expansions of non holomorphic deformations of solutions of wave or elasticity equations. They are central for the study of the complex moment problem. We can also mention the study of real subvarieties of $\mathbf{C}^2$, among others by Moser and his collaborators. Harmonic polynomial and rational maps are important special cases of mixed polynomials ; they have been extensively studied and were applied to the study of gravitational lensing [KN05].

Several techniques developed in Computer algebra are useful for understanding these objects. We revisit, from an algorithmic point of view, the roots study of pairs of real bivariate polynomials (f, g). The case $m = 1$ could be called "almost holomorphic", and we look for properties similar to those of "usual" univariate polynomials. Moreover after simplification it reduces to the study of $\bar{z} = r(z)$, where r is a rational rational map : we will briefly recall recent advances obtained in that field, [3, 5, 1].

One of our tool will be a variant of Vandermonde matrix that we will use to interpolate $P(z, \bar{z})$. Similarly, we will specify a set of roots in $\mathbf{C}$ and investigate the maximum number of other roots in $\mathbf{C}$ admitted by such a constrained mixed polynomial. Unfortunately, the presentation of a univariate polynomial as a product via its roots is not valid in this context. As we will see, although P of bidegree $(n, 1)$ has $2n + 2$ coefficients, it may admit more than $2n + 2$ roots in $\mathbf{C}$. We will discuss and illustrate this behavior, directly related to bounding the number of zeros of harmonic maps. Beside the case $m = 1$, the results obtained so far on harmonic polynomials, concentrated on m near n, while we are more attracted by small m, see [2]. We will also describe, in small degrees, the partition in semi-algebraic cells of the coefficient spaces corresponding to a given number of roots : their shapes resemble to domains delimited by the generalized "swallow tails" used by R. Thom in his Catastrophes theory.

Another objects of interest are the mixed polynomials, of degrees $(n, 1)$, with given random distribution of coefficients. Experiments with the computer algebra system Maple allowed to observe interesting patterns.

Références

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