Semidefinite approximations of projections and polynomial images of semialgebraic sets

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Given a compact semialgebraic set $\mathbf{S} \subseteq \mathbb{R}^n$, a polynomial map $f : \mathbf{S} \to \mathbb{R}^m$, we consider the problem of approximating the image set $\mathbf{F} = f(\mathbf{S})$. This includes in particular the projections of $\mathbf{S}$ on $\mathbb{R}^m$, for $n \geq m$. Assuming that $\mathbf{F} \subseteq \mathbf{B}$, with $\mathbf{B} \subseteq \mathbb{R}^m$ being a "simple" set (box or ellipsoid), we provide two methods (called Method 1 and Method 2) to compute certified outer approximations of $\mathbf{F}$:

— The first approach (Method 1) consists in rewriting $\mathbf{F}$ as a set defined with an existential quantifier. Then, one can outer approximate $\mathbf{F}$ as closely as desired with a hierarchy of superlevel sets of the form $\mathbf{F}_r^1 := \{\mathbf{y} \in \mathbf{B} : q_r(\mathbf{y}) \geq 0\}$, for some polynomials $q_r \in \mathbb{R}[\mathbf{y}]$ of increasing degrees $2r$.

— The second approach (Method 2) consists in building a hierarchy of relaxations for the infinite dimensional moment problem whose optimal value is the volume of $\mathbf{F}$ and whose optimum is the restriction of the Lebesgue measure on $\mathbf{F}$. Then, one can outer approximate $\mathbf{F}$ as closely as desired with a hierarchy of super level sets of the form $\mathbf{F}_r^2 := \{\mathbf{y} \in \mathbf{B} : w_r(\mathbf{y}) \geq 1\}$, for some polynomials $w_r \in \mathbb{R}[\mathbf{y}]$ of increasing degrees $2r$.

These two methods output a sequence of superlevel sets defined with a single polynomial that yield explicit outer approximations of $\mathbf{F}$. Finding the coefficients of this polynomial boils down to compute an optimal solution of a semidefinite program. We provide guarantees of strong convergence to $\mathbf{F}$ in $L_1(\mathbf{B})$-norm, when the degree of the polynomial approximation tends to infinity.

We next present some application examples together with numerical results. In particular, we illustrate that our methodology is a unified framework which can tackle important special cases : semialgebraic set projections and Pareto curves approximations. The framework can be extended to approximate images of semialgebraic sets under semialgebraic applications.