Automatic Continued Fractions Expansions by *Guess and Prove*

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Continued Fractions. Continued fractions have been used since Euler's time for their remarkable convergence properties [Eul48]. Among the two-dimensional Padé table formed by the rational approximants P/Q to a given complex series, they form a diagonal staircase. Restricting the approximation to the diagonal is usually preferred because it is formally simple to compute.

The analytical and numerical properties of continued fractions have been studied extensively since the 60's, as can be seen in numerous reference books on the topic (*e.g.* [JT80]), and we refer to Brezinski for a historical point of view [Bre81]. As an example, the following formal expansion of the natural logarithm as a so-called *C-fraction* provides an analytic continuation to the whole complex plane cut along the negative real axis :

$$\ln(1+z) = \cfrac{z}{1+\cfrac{a_2 z}{1+\cfrac{a_3 z}{1+\dots}}} \text{ for all } z,\ (1+z) \in \mathbb{C}\setminus\mathbb{R}_-.$$

where $a_{2k} = \frac{k}{2(2k-1)}$ and $a_{2k+1} = \frac{k}{2(2k+1)}$. Compared to the Taylor series, it not only has a wider convergence domain, but also converges faster on the disk $|z| < 1$.

Automation. In many similar cases of interest, simple formulas can be derived for the coefficients of a continued fraction of this shape. As can be seen in a recent compendium on the topic by Cuyt *et alii* [CPV+08], most expansions are obtained by hand, by specializing a number of formulas. We propose here to use data structures from computer algebra, and proof techniques from experimental mathematics, to obtain such formulas automatically, in a unified manner.

Given a power series, our procedure provides a way of :
- detecting instantly if the coefficients of its C-fraction expansion may satisfy a small-order recurrence,
- computing a simple proof for the formal correspondence of this (infinite) expansion to the input series.

Notably, the proofs are obtained in a generic way, using a single procedure, which contrasts with the limits taken coefficient by coefficient in the literature. As such, this work can also be seen as a new direct proof of the expansion for the exponential function for example [Wal48].

The proof is performed using the very general framework of holonomic series as the underlying function representation.

Under the hood. *Holonomic* functions ([Sta80]), cover a wide class of the so–called *special functions* from mathematical physics, combinatorics, *etc.*. It consists of functions which can be implicitely represented using a linear differential equation with polynomial coefficients, along with initial conditions. Equivalently, a holonomic sequence (of coefficients) is represented using a linear recurrence relation with polynomial coefficients. The efficient implementation of these objects and operations in the maple module gfun [SZ94] served as a basis for experimentation and development, to provide reactive tools.

Among others properties, the class of holonomic functions enjoys an algorithmic ring structure. Interestingly, the fact that division does not preserve holonomicity is not an issue here, thanks to a natural approach in the holonomic world : "guess and prove".

Guess and prove. At first, only a finite order expansion is known, providing say the 30 first terms of the continued fraction. A recurrence on its first coefficients can be computed using standard linear algebra — this is the "guessing" step. In a considerable number of examples, the recurrence order is strikingly small (≤ 3).

The "guessed" recurrence then provides a description of an infinite continued fraction, which must be proved equal to the original function. This verification step first involves simple formulas concerning continued fractions, and the algorithmic closure properties of holonomic sequences. But more crucially, another "guess and prove" step is needed, in order to check the differential equation on the conjectured expansion. This last part is the most time-consuming part of the proof, and necessitated optimizations here.

Bibliographie

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