Computing real points on determinantal varieties and spectrahedra

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October 20, 2014

Introduction. We are interested in the geometry of real algebraic varieties defined by rank constraints on square matrices whose entries are linear forms with rational coefficients:

$$\mathcal{D} = \Big\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : A(x) = A_0 + x_1 A_1 + \ldots + x_n A_n \text{ has rank } \leq r \Big\}$$

given integers m, n, r and $A_i \in \mathbb{Q}^{m\times m}$ for $i = 0, \ldots, n$. Sets of this type are defined by collections of minors of A and are called *real determinantal varieties*. They are ubiquitous in the mathematical sciences and in applications. If $A_0, \ldots, A_n$ lie in some linear subspace of $\mathbb{Q}^{m\times m}$, so does $A(x)$. In particular, if they are symmetric, then the set

$$\mathcal{S} = \Big\{ x \in \mathbb{R}^n : A(x) \text{ is positive semi-definite} \Big\},$$

provided it is full-dimensional, is called the *spectrahedron* associated to A. Spectrahedra are affine sections of the cone of positive semi-definite matrices, and also convex basic semi-algebraic sets (for example, polyhedra are particular examples of spectrahedra, when all the matrices A_i commute, and in particular if they are all diagonal). These are the feasible sets of *semidefinite programming*, whose goal is to minimize linear functions over $\mathcal{S}$. Now, solutions to semidefinite programs are algebraic points lying in the boundary of the set $\mathcal{S}$, which is a subset of the hypersurface defined by $\det A(x) = 0$. In general, the matrix A has rank defects at all points of the boundary of $\mathcal{S}$: this provides a geometric relation between the stratifications of the rank of A and the set $\mathcal{S}$. Hence, it is a problem of primary importance to design exact algorithms solving efficiently what follows:

- decide whether $\mathcal{S}$ is empty or not;
- compute the smallest rank atteined by $A(x)$ on $\mathcal{S}$;
- compute a point on the boundary of $\mathcal{S}$ where the smallest rank is attained.

Also, in some applications, the matrix $A(x)$ belongs to some fixed subspace of $\mathbb{Q}^{m\times m}$, for example the space of Hankel or Hurwitz matrices. We also consider these *structured* situations which often occur and are interesting in different areas.

Contributions. Our main contribution is the construction of an exact algorithm for finding at least one point in every connected component of the set $\mathcal{D}$. This is a particular instance of the general problem of solving systems of polynomial equations over the real numbers and represents a possible strategy to decide the emptiness of such sets: in fact, if the rank of $A(x)$ is at most r at some real point x, then $\mathcal{D}$ is non-empty and the algorithm is expected to give as output a representation of a finite set of points containing at least one point per connected component of the real set; otherwise it returns the empty set.

Under genericity assumptions on the entries of $A_0, \ldots, A_n$, the aforementioned algorithm produces a rational parametrization of a finite set intersecting each connected components of $\mathcal{D}$; in case of success its runtime is essentially quadratic on a multihomogeneous Bézout bound on the number of complex solutions, which is strictly upper bounded by $\binom{m(m-r)+n}{n}^3$. In particular, when the size of the matrix is fixed, the complexity is at most polynomial in the number of variables. Moreover, it has a good asymptotic behavior (when both m and n go to infinity). This improves the state of the art since algorithms solving this problem typically require (at most) $d^{\mathcal{O}(N)}$ arithmetic operations when dealing with a polynomial equation of degree d in N variables. This improvement arises from the particular nature of the polynomial system under study, and we will also discuss numerical results supporting this theoretical complexity gain.

The interesting fact is that if the linear matrix has a structure in the sense mentioned above, the bounds on the number of solutions computed by the algorithm are significantly smaller, and the same holds for the computational complexity. For example, the previous Bézout bounds for affine sections of symmetric matrices and Hankel matrices are respectively $\binom{(m-r)(m+r+1)/2+n}{n}^3$ and $\binom{2m-r-1+n}{n}^3$.

Finally, this algorithm can be used to compute points lying on the boundary of a given spectrahedron. This is possible since, in the symmetric case, the boundary of the spectrahedron $\mathcal{S}$ of $A(x)$ contains a connected component of the determinantal variety $\mathcal{D}$ where r is the minimum possible rank appearing on $\mathcal{S}$. This result proves that our algorithm computes a small-rank point lying on the boundary of $\mathcal{S}$ (if $\mathcal{S} \neq \emptyset$) and that it can answer the three questions mentioned above. The complexity of this problem is also a polynomial function of the aforementioned multihomogeneous Bézout bound on the number of complex solutions for symmetric linear matrices. This fact is remarkable because we can derive a complexity estimate for the problem of deciding the emptyness of spectrahedra.