Harmonic sums and polylogarithms at non-positive integers

Gerard.H.E. Duchamp, Vincel. Hoang Ngoc Minh, **Quoc Hoan. Ngo**
Universite Paris 13, LIPN
99 avenue Jean-Baptiste Clement, 93430 Villetaneuse.

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We are interested in the implementation in Maple of the following objets

$$H_{-s_1,..,-s_r}(N) = \sum_{N\geq n_1>...>n_r>0} n_1^{s_1}...n_r^{s_r}, \forall N \in \mathbb{N}_+, \tag{1}$$

$$Li_{-s_1,...,-s_r}(z) = \sum_{n_1>...>n_r>0} z^{n_1} n_1^{s_1}...n_r^{s_r}, \forall z \in \mathbb{C}, \tag{2}$$

$$\zeta(-s_1,...,-s_k) = \sum_{n_1=0}^{\infty} ... \sum_{n_k=0}^{\infty} (1+n_1)^{s_1} ... (k+n_1+...+n_k)^{s_k}, \tag{3}$$

where $s_1,...,s_k$ are non-negative integers. [1].

Precisely,

- We prove that $H_{-s_1,..,-s_r}(N)$ is a polynomial of degree $s_1+...+s_r+r$ of N and we give the explicit formula to compute the constants $C_{-s_1,...,-s_r}$ such that
$$\lim_{N\to\infty} \frac{C_{-s_1,...,-s_r} N^{s_1+...+s_r+r}}{H_{-s_1,..,-s_r}(N)} = 1. \tag{4}$$
- We prove also that $Li_{-s_1,...,-s_r}(z)$ is a polynomial of degree $s_1+...+s_r+r$ of $(1-z)^{-1}$ and we give the explicit formula to compute the constants $B_{s_1,...,s_r}$ such that
$$\lim_{z\to 1^-} \frac{B_{s_1,...,s_r}(1-z)^{-(s_1+...+s_r+r)}}{Li_{s_1,..,s_r}(z)} = 1. \tag{5}$$

1. Quantities in eq. 3 are divergent, the aim of this work is to give tools to approach these divergences.

– We give the shuffle structure for (1).
– We study the values of (3) at negative integers, by analytic prolongation.

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