# Algorithme d'optimisation polynomiale en utilisant de bases de bord

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# Our Problem

$$f^* = inf_{\mathbf{x} \in \mathcal{S}} f(\mathbf{x})$$

where $\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^n \mid g_1^0(\mathbf{x}) = \cdots = g_{n_1}^0(\mathbf{x}) = 0, g_1^+(\mathbf{x}) \geq 0, ..., g_{n_2}^+(\mathbf{x}) \geq 0\}$

and $V_{min} = \{\mathbf{x}^* \in \mathcal{S} \ s.t \ f(x^*) = f^*\}$

# Moment problem

$f^* = inf_{\mathbf{x} \in \mathcal{S}} f(\mathbf{x})$ where $f(\mathbf{x}) = \sum f_\alpha \mathbf{x}^\alpha$

$f^* = inf_\mu \int_\mathcal{S} f(\mathbf{x})\mu(dx) = inf_\mu \sum f_\alpha \int_\mathcal{S} \mathbf{x}^\alpha \mu(dx)$ where $\int_\mathcal{S} \mathbf{x}^\alpha \mu(dx) = \mathbf{y}_\alpha$

$f^* = inf \;\; f^T \mathbf{y}$ s.t $\mathbf{y}_0 = 1$, $\mathbf{y}$ has a representating measure on S

# Lasserre Full moment relaxation

$$f_G^{\mu} = inf\ \Lambda(f) \text{ s.t } \Lambda(1) = 1,\ \Lambda(p) \geq 0\ \forall p \in \overbrace{\sum_{g \in G^0} g\, h + \sum_{g' \in \prod G^+} g'\, h' \ with\ h \in \mathbb{R}[\mathbf{x}], h'\ is\ sos}^{\mathcal{P}_G}.$$

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IDEA-Truncated convex problems relaxation

$$f_{t,G}^{\mu} = inf\ \{\Lambda(f)\ s.t.\ \Lambda \in \mathbb{R}[\mathbf{x}]_{2t}^{*},\ \Lambda(1) = 1,\ \Lambda(p) \geq 0,\ \forall p \in \mathcal{P}_{t,G}\}$$

where

$$\mathcal{P}_{t,G} = \left\{ \sum_{i=1}^{n_1} h g_i^0 + \sum_{\nu \in \{0,1\}^m} h' g_1^+(\mathbf{x})^{\nu_1} \cdots g_{n_2}^+(\mathbf{x})^{\nu_{n_2}} |\ \ h'\ sos\ degree\ t - \left\lceil \frac{\prod g_i^+}{2} \right\rceil, h \in \mathbb{R}[\mathbf{x}]_{2t - deg(g_i^0)} \right\}$$

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$$\cdots f_{t,G}^{\mu} \leq f_{t+1,G}^{\mu} \leq \cdots \leq f^*$$

# Why a new algorithm using border basis?

**Issues of Lasserre relaxation**:
The size and number of parameters in each problem become unhandled when t grows.

**Using border basis**

- We reduce the size and number of parameters of each problem.
- Sometimes we find the minimum at a lower degree.

# Border basis

We take $B$ a monomial set **connected to 1** and we define $B^+ = B \cup x_1 B \cup \cdots \cup x_n B$ and $\partial B = B^+ \setminus B$.

## Definition

*$F \subset \mathbb{R}[\mathbf{x}]$ is a **border basis** for $B$ in degree $t \in \mathbb{N}$, if*

- $\forall m \in \partial B \cap \mathbb{R}[\mathbf{x}]_t$, $\exists f \in F_t$ s.t. $f = m + \sum_{b \in B_t} c_i b$
- $\mathbb{R}[\mathbf{x}]_t = \langle B \rangle_t \oplus \langle F | t \rangle$

## Definition

*We define $\pi_{B_t, F_{2t}}$ the **projection** of $\mathbb{R}[\mathbf{x}]_{2t}$ on $\langle B_t \rangle$ along $\langle F \mid 2t \rangle$.*

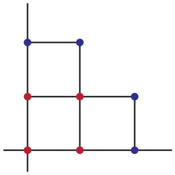
# Border basis: example

$B = \{1, x, y, xy\}$,

$B^+ = B \cup xB \cup yB =$
$\{1, x, y, xy, x^2, x^2y, xy^2, y^2\}$,

$\partial B = B^+ \setminus B = \{x^2, x^2y, xy^2, y^2\}$

$F_2 = \{x^2 - 1, y^2 - 1\}$



# Border basis relaxation

## Border basis hierarchy

- *$F_{2t}$ border basis of $g^0$ for $B$ in degree $2t$,*
- $G_t^0 = \{p - \pi_{B_t,F_{2t}}(p),\ p \in B_t \cdot B_t\}$
- $G_t^+ = \pi_{B_t,F_{2t}}(\mathbf{g}^+)$.

## Proposition

$$\cdots \leq f^{\mu}_{B_t,G_t} \leq f^{\mu}_{B_{t+1},G_{t+1}} \leq \cdots \leq f^*$$

# Truncated Hankel operators and Optimal linear form

## Definition

- *Let $E \subset \mathbb{R}[\mathbf{x}]$ be a vector space and let $\Lambda \in \langle E \cdot E\rangle^*$ be a linear form,*

$$\begin{array}{rcl} H_\Lambda^E : E & \longrightarrow & E^* \\ p & \longmapsto & p \cdot \Lambda = \Lambda(pq) \ \forall q \in E \end{array}$$

  *is a truncated Hankel operator.*
- $\ker H_\Lambda^E = \{p \in E \mid p \cdot \Lambda = 0, i.e,\ \Lambda(pq) = 0 \ \forall q \in E\}$
- *The matrix of this operator $H_\Lambda^E$ is the* ***Moment Matrix****:* $[H_\Lambda^E] = (\Lambda(\mathbf{x}^{\alpha+\beta}))_{x^\alpha \in E, x^\beta \in E}$

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$$\mathcal{L}_{E,G} = \{\Lambda \in \langle E \cdot E \rangle^*,\ \Lambda(1) = 1,\ \Lambda(p) \geq 0,\ \forall p \in \mathcal{P}_{E,G}\}$$

## Definition

$\Lambda^* \in \mathcal{L}_{E,G}$ *is optimal linear form for f if rank* $H_{\Lambda^*}^E = \max_{\Lambda \in \mathcal{L}_{E,G}, \Lambda(f) = f^\mu_{E,G}}$ *rank* $H_\Lambda^E$.

## Theorem

*Let our polynomial f and the set $\mathcal{S}$ and let $G \subset \mathbb{R}[\mathbf{x}]$ be a set of constraints defined as our border basis relaxation. There exist $t_1 \in \mathbb{N}$ such that $\forall t \geq t_1$*

- $f^{\mu}_{t,G} = f^*$ *is reached for some* $\Lambda^* \in \mathcal{L}_{t,G}$
- $\forall\ \Lambda^* \in \mathcal{L}_{t,G}$ *optimal for f* , $\Lambda^*(f) = f^*$ *and* $\ker H^t_{\Lambda^*} = I_{min}$.

How to find $\Lambda^* \in \mathcal{L}_{E,G}$ optimal linear form for $f$?

# Optimal linear form Algorithm- SemiDefinite Programming

**Input:** $f \in \mathbb{R}[\mathbf{x}]$, $B_t = (\mathbf{x}^{\alpha})_{\alpha \in A}$ a monomial set containing 1 with $f \in \langle B_t \cdot B_t \rangle$, $G \subset \mathbb{R}[\mathbf{x}]$.

- SDP problem: $argmin_{\Lambda}\ \Lambda(f)$ s.t.
  1. $H_{\Lambda^*}^{B_t} = (\Lambda^*(\mathbf{x}^{\alpha+\beta}))_{\alpha,\beta \in A} \succcurlyeq 0$, and $\Lambda^*(1) = 1$
  2. $H_{g^+ \cdot \Lambda^*}^{B_{t-w}} \succcurlyeq 0$ for all $g^+ \in G^+$ where $w = \lceil \frac{deg(g^+)}{2} \rceil$.
  3. $\Lambda^*(g^0) = 0\ \forall g^0 \in G^0 \cap \langle B_t \cdot B_t \rangle$.

**Output:** the minimum $f_{t,G}^{\mu}$ of $f$ and $\Lambda^* \in \langle B_t \cdot B_t \rangle^*$.

☞ Tools for solving this SemiDefinite Programing problem: SDPA, MOSEK,...

# Example for $t = 3$:

Minimize $x^2 + 3$ s.t. $g = x^4 - x^3 - x + 1 = (x-1)^2(x^2+x+1) = 0$.

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Minimize $\Lambda(x^2 + 3) = \Lambda(x^2) + 3$
with $\Lambda$ s.t.
$\Lambda(1) = 1$,
$\Lambda(x^4) = \Lambda(x^3) + \Lambda(x) - \Lambda(1)$,
$\Lambda(x^5) = \Lambda(x^3) + \Lambda(x^2) - \Lambda(1)$,
$\Lambda(x^6) = 2\,\Lambda(x^3) - \Lambda(1)$
and

$$H_\Lambda^3 := \begin{pmatrix} 1 & a & b & c \\ a & b & c & c+a-1 \\ b & c & c+a-1 & c+b-1 \\ c & c+a-1 & c+b-1 & 2\,c-1 \end{pmatrix} \succcurlyeq 0$$

where $a = \Lambda(x)$, $b = \Lambda(x^2)$, $c = \Lambda(x^3)$.

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where $a = \Lambda(x)$, $b = \Lambda(x^2)$, $c = \Lambda(x^3)$.

Solution: $\Lambda^*(1) = 1$, $\Lambda^*(x) = 1$, $\Lambda^*(x^2) = 1$, $\Lambda^*(x^3) = 1$ ... and the minimum is $\Lambda(x^2 + 3) = \Lambda(x^2) + 3 = 4$.

$\ker H_{\Lambda^*} = \langle x - 1, x^2 - x, x^3 - x^2 \rangle$.

**How to detect when the minimum is reached?**$(\Lambda^*(f) = f^*)$

**(Curto-Fialkov 96) If** $H_\Lambda^E \succcurlyeq 0$ **and** $rank\ H_\Lambda^E = r$ **then**

$$\Lambda = \sum_{i=1}^{r} \omega_i \mathbf{1}_{\xi_i},\ \ \xi_i \in V(ker\ H_\Lambda^E)$$

# Convergence certificate

## Theorem - Convergence Certification

*Let* $B \subset E \subset \mathbb{R}[\mathbf{x}]$ *be finite dimensional vector spaces connected to* 1 *with*

- $B^+ \subset E$,
- $G^0 \cdot B \subset \langle E \cdot E \rangle$,
- $G^+ \cdot B \cdot B \subset \langle E \cdot E \rangle$.

*Let* $\Lambda \in \mathcal{L}_{E,G}$ *with* $\boxed{\text{rank } H_\Lambda^E = \text{rank } H_\Lambda^B = \dim B}$. *Then*

1. $\exists\, \tilde{\Lambda} \in \mathbb{R}[\mathbf{x}]^*$ *which extends* $\Lambda$,
2. $\tilde{\Lambda} = \sum_{i=1}^{r} \omega_i \mathbf{1}_{\xi_i}$ *with* $\omega_i > 0, \xi_i \in \mathcal{S}(G)$,
3. $(\ker H_\Lambda^E) = I(\xi_1, \ldots, \xi_r)$.

# How to check this flat extension property? Flat extension Algorithm

**Input:** a vector space $E$ connected to 1 and a linear form $\Lambda^* \in \langle E \cdot E \rangle^*$ and the inner product $\langle p, q \rangle_* := \Lambda^*(p\, q)$.

1. Take $B_0 := \{1\}$, $i = 0$;
2. While $B_i^+ \subset E$ or $L_i \neq \{0\}$
   - compute $L_i$ of maximal dim. in $B_i^+$
     - $\langle L_i, B_i \rangle_* = 0$
     - $L_i \cap \ker H_{\Lambda^*}^{B_i^+} = \{0\}$.
   - $B_{i+1} = B_i + L_i$, $i = i + 1$.
3. If $B_i^+ \not\subset E \Rightarrow$ `failed`

   else $L_i = \{0\}$ and $B_i^+ = B_i \oplus \ker H_{\Lambda^*}^{B_i^+} \Rightarrow$ `success`.

**Output:** *failed* or *success* with

- a basis $B = \{b_1, \ldots, b_r\} \subset \mathbb{R}[\mathbf{x}]$;
- the relations $x_k b_j - \sum_{i=1}^r \frac{\langle x_k b_j, b_i \rangle_*}{\langle b_i, b_i \rangle_*} b_i$, $j = 1 \ldots r$ $k = 1 \ldots n$.

# Flat extension Algorithm: Example

$min\ f(x) = (x-1)^2 \cdot (x-2)^2 \cdot (x^2+1) + (y-1)^2 \cdot (y^2+1)$

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We compute the optimal linear form (SDP with t=3) and we obtain:
$\Lambda^*(1) = 1, \Lambda^*(x) = 1.5, \Lambda^*(y) = 1, \Lambda^*(x^2) = 2.5, \Lambda^*(xy) = 1.5, \Lambda^*(y^2) = 1, \Lambda^*(y^3) = 1, \Lambda^*(xy^2) = 1.5, \Lambda^*(x^2y) = 2.5, \Lambda^*(x^3) = 4.5, \Lambda^*(y^4) = 1, \Lambda^*(xy^3) = 1.5, \Lambda^*(x^2y^2) = 2.5, \Lambda^*(x^3y^) = 4.5, \Lambda^*(x^4) = 8.5, \ldots$

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- $B_0 = \{1\},\ \partial B_0 = \{x, y\},\ B_0^+ = \{1, x, y\}$

$$H_\Lambda^{B_0^+} = \begin{pmatrix} 1 & 1.5 & 1 \\ 1.5 & 2.5 & 1.5 \\ 1 & 1.5 & 1 \end{pmatrix} \longrightarrow H_\Lambda^{\{1, x-1.5, y-1\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow rank\ H_\Lambda^{B_0^+} = 2,$$

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- $B_1 = B_0 + L_0 = \{1, x-1.5\},\ \partial B_1 = \{y, x^2-1.5x, xy-1.5y\},\ B_1^+ = \{1, x-1.5, y, x^2-1.5x, xy-1.5y\}$

$$H_\Lambda^{B_1^+} = \begin{pmatrix} 1 & 0 & 1 & 0.25 & 0 \\ 0 & 0.25 & 0 & 0.375 & 0.25 \\ 1 & 0 & 1 & 0.25 & 0 \\ 0.25 & 0.375 & 0.25 & 0.625 & 0.375 \\ 0 & 0.25 & 0 & 0.375 & 0.25 \end{pmatrix} \longrightarrow H_\Lambda^{\tilde{B}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $\tilde{B} = \{1, x-1.5, y-1, x^2-3x+2, xy-1.5y-x+1.5\}$

$$\longrightarrow rank\ H_\Lambda^{B_1^+} = 2, L_1 = \{0\},$$

# Flat extension Algorithm: Example

$$\textit{min } f(x) = (x-1)^2 \cdot (x-2)^2 \cdot (x^2+1) + (y-1)^2 \cdot (y^2+1)$$

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- $B_0 = \{1\},\ \partial B_0 = \{x, y\},\ B_0^+ = \{1, x, y\}$

$$H_\Lambda^{B_0^+} = \begin{pmatrix} 1 & 1.5 & 1 \\ 1.5 & 2.5 & 1.5 \\ 1 & 1.5 & 1 \end{pmatrix} \longrightarrow H_\Lambda^{\{1, x-1.5, y-1\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \textit{rank } H_\Lambda^{B_0^+} = 2,$$

- $B_1 = B_0 + L_0 = \{1, x-1.5\},\ \partial B_1 = \{y, x^2 - 1.5x, xy - 1.5y\},\ B_1^+ = \{1, x-1.5, y, x^2 - 1.5x, xy - 1.5y\}$

$$H_\Lambda^{B_1^+} = \begin{pmatrix} 1 & 0 & 1 & 0.25 & 0 \\ 0 & 0.25 & 0 & 0.375 & 0.25 \\ 1 & 0 & 1 & 0.25 & 0 \\ 0.25 & 0.375 & 0.25 & 0.625 & 0.375 \\ 0 & 0.25 & 0 & 0.375 & 0.25 \end{pmatrix} \longrightarrow H_\Lambda^{\tilde{B}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $\tilde{B} = \{1, x-1.5, y-1, x^2-3x+2, xy-1.5y-x+1.5\}$

$\longrightarrow \textit{rank } H_\Lambda^{B_1^+} = 2, L_1 = \{0\},$
$\textit{Ker } H_\Lambda^{B_1^+} = \{y-1, x^2-3x+2, xy-1.5y-x+1.5\}.$

# How to compute minimizers points?
## Minimizer points Algorithm

**Multiplication operators**: $M_k : a \in \mathcal{A}_{min} \mapsto x_k a \in \mathcal{A}_{min}$ $k = 1, \ldots, n$ where $\mathcal{A}_{min} = \mathbb{R}[\mathbf{x}]/I_{min}$ and $I_{min} = \mathcal{I}(\xi_1, \ldots, \xi_r) = \langle x_k b_j - \sum_{i=1}^{r} \frac{\Lambda^*(x_k b_j b_i)}{\Lambda^*(b_i b_i)} b_i \rangle_{j=1\ldots r, k=1\ldots n}$.

**Input:** $B = \{b_1, \ldots, b_r\}$ basis of $\mathcal{A}_{min}$ and $x_k b_j \equiv \sum_{i=1}^{r} \frac{\Lambda^*(x_k b_j b_i)}{\Lambda^*(b_i b_i)} b_i \ mod \ I_{min}$

- Compute the matrices of the operator $M_k$ in the basis $B$: $[M_k] = (\frac{\Lambda^*(x_k \ b_i \ b_j)}{\Lambda^*(b_i \ b_i)})_{1 \leq i,j \leq r}$.
- For a generic choice of $l_1, \ldots, l_n \in \mathbb{R}$, compute the eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_r$ of $l_1[M_1] + \cdots + l_n[M_n]$.
- Compute $\xi_{i,k} \in \mathbb{R}$ such that $M_k \mathbf{u}_i = \xi_{i,k} \mathbf{u}_i$.

**Output:** the minimizers $\xi_i = (\xi_{i,1}, \ldots, \xi_{i,n})$, $i = 1 \ldots r$.

# Minimizer points Algorithm: Example

$$M_x^{B=\{1,x-1.5\}} = \begin{pmatrix} 1.5 & 0.25 \\ 1 & 1.5 \end{pmatrix} \longrightarrow \begin{cases} x = 1.5 \cdot 1 + 1 \cdot (x - 1.5) \\ x(x-1.5) = 0.25 \cdot 1 + 1.5 \cdot (x - 1.5) \end{cases}$$

$$M_y^{B=\{1,x-1.5\}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow \begin{cases} y = 1 \cdot 1 + 0 \cdot (x - 1.5) \\ y(x-1.5) = 0 \cdot 1 + 1 \cdot (x - 1.5) \end{cases}$$

$$M = M_x^B + M_y^B = \begin{pmatrix} 2.5 & 0.25 \\ 1 & 1.5 \end{pmatrix} \longrightarrow \lambda_1 = 2, \lambda_2 = 3$$

$$M \cdot u_1 = \lambda_1 \cdot u_1 \rightarrow u_1^T = (-0.5, 1);\ M \cdot u_2 = \lambda_2 \cdot u_2 \rightarrow u_2^T = (0.5, 1)$$

$$M_x^B \cdot u_1^T = x_1 \cdot u_1^T \rightarrow x_1 = 1;\ M_x^B \cdot u_2^T = x_2 \cdot u_2^T \rightarrow x_2 = 2$$

$$M_y^B \cdot u_1^T = y_1 \cdot u_1^T \rightarrow y_1 = 1;\ M_y^B \cdot u_2^T = y_2 \cdot u_2^T \rightarrow y_2 = 1$$

The minimizers points are $(1,1)$ and $(2,1)$.

# Minimizer Border Basis Algorithm

**Input:** A real polynomial function $f$ and a set of constraints $\mathbf{g} \subset \mathbb{R}[\mathbf{x}]$ with $V_{min}$ non-empty finite.

1. $t = max(\lceil \frac{deg(f)}{2} \rceil, d^0, d^+)$ where $d^0 = max_{g^0 \in \mathbf{g}^0}(\lceil \frac{deg(g^0)}{2} \rceil), d^+ = max_{g^+ \in \mathbf{g}^+}(\lceil \frac{deg(g^+)}{2} \rceil)$
2. Compute the graded border basis $F_{2t}$ of $\mathbf{g}^0$ for B in degree $2t$.
3. Let $B_t$ be the set of monomials in $B$ of degree $\leq t$.
4. Let $G_t$ be the set of constraints such that $G_t^0 = \{m - \pi_{B_t, F_{2t}}(m),\ m \in B_t \cdot B_t\}$ and $G^+ = \pi_{B_t, F_{2t}}(g^+)$
5. $[f^*_{G_t, B_t}, \Lambda^*] :=$ OPTIMAL LINEAR FORM$(f, B_t, G_t)$.
6. $[c, B', K] :=$ FLAT EXTENSION$(\Lambda^*, B_t)$
7. if $c =$`fails` then go to step 2

   else with $t := t + 1$. $c =$`success` then $V=$ MINIMIZER POINTS $(B', K)$

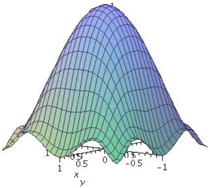
**Output:** the minimum $f^* = f^*_{G_t, B_t}$, the minimizers $V_{min} = V$, $I_{min} = (K)$.

# Example - Robinson polynomial

$f(x,y) = 1 + x^6 - x^4 - x^2 + y^6 - y^4 - y^2 - x^4y^2 - x^2y^4 + 3x^2y^2;$

$\nabla f = (6x^5 - 4x^3 - 2x - 4x^3y^2 - 2xy^4 + 6xy^2, 6y^5 - 4y^3 - 2y - 4y^3x^2 - 2yx^4 + 6yx^2)$

| Iteration | Degree | Size | Parameters | min $\Lambda(f)$ | Flat extension |
|---|---|---|---|---|---|
| 1 | 3 | 10 | 21(28) | -0.93 | failed |
| 2 | 4 | 15 | 21(45) | 0 | success |




We obtain:

- $f^* = 0$;
- $I_{min} = (x^3 - x, y^3 - y, x^2y^2 - x^2 - y^2 + 1)$
- $B = \{1, x, y, x^2, xy, y^2, x^2y, xy^2\}$
- $V_{min} = \{(1,1), (1,-1), (-1,1), (-1,-1), (1,0), (-1,0), (0,1), (0,-1)\}$

Note: Gloptipoly must go to degree $t = 7$ to find the minimum.

# Implementation and comparasion with Gloptipoly

- Implementation
  - This algorithm is implemented in the package BORDERBASIX of MATHEMAGIX software.
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  - If there are equality constraints the size of the moments matrices and the number of parameters in our method is smaller due to the use of border basis.
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# Experimentations

| v | c | d | sol | $t_{bbr}$ | $o_{bbr}$ | $p_{bbr}$ | $s_{bbr}$ | $t_{bbr+msk}$ | $t_{fmr}$ | $o_{fmr}$ | $p_{fmr}$ | $s_{fmr}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 0 | 6 | 8 | 0.15 | 4 | 21 | 15 | 0.08 | * | 119 | 36 | |
| 2 | 0 | 6 | 4 | 0.17 | 4 | 26 | 15 | 0.12 | * | 9 | 189 | 55 |
| 3 | 1 | 6 | 1 | 3.78 | 5 | 167 | 56 | 1.27 | 9.57 | 5 | 286 | 56 |
| 2 | 0 | 4 | 1 | 0.030 | 2 | 8 | 6 | 0.023 | 0.050 | 2 | 14 | 6 |
| 2 | 0 | 4 | 1 | 0.030 | 2 | 8 | 6 | 0.028 | 0.050 | 2 | 14 | 6 |
| 2 | 0 | 6 | 4 | 0.432 | 4 | 25 | 15 | 0.091 | * | 8 | 152 | 45 |
| 2 | 3 | 2 | 3 | 0.045 | 2 | 14 | 6 | 0.035 | 0.053 | 2 | 14 | 6 |
| 1 | 2 | 4 | 1 | 0.023 | 2 | 4 | 3 | 0.022 | 0.042 | 2 | 4 | 3 |
| 2 | 5 | 4 | 1 | 0.060 | 2 | 13 | 6 | 0.03 | 0.077 | 2 | 14 | 6 |
| 2 | 6 | 4 | 1 | 0.20 | 4 | 44 | 15 | 0.12 | 0.29 | 4 | 44 | 15 |
| 5 | 11 | 2 | 1 | 7.60 | 3 | 461 | 56 | 4.45 | 12.23 | 3 | 461 | 56 |
| 6 | 13 | 2 | 1 | 1.00 | 2 | 209 | 26 | 0.40 | 1.29 | 2 | 209 | 26 |
| 6 | 15 | 2 | 1 | 1.01 | 2 | 209 | 26 | 0.42 | 1.48 | 2 | 209 | 26 |
| 10 | 11 | 2 | 1 | 12.3 | 2 | 714 | 44 | 1.83 | 16.76 | 2 | 1000 | 55 |
| 10 | 25 | 2 | 1 | 28.60 | 2 | 1000 | 66 | 5.92 | 43.68 | 2 | 1000 | 66 |
| 10 | 31 | 2 | 1 | 29.70 | 2 | 1000 | 66 | 12.50 | 44.29 | 2 | 1000 | 66 |
| 13 | 35 | 2 | 1 | 383.97 | 2 | 2379 | 78 | 32.41 | 417.96 | 2 | 2379 | 78 |
| 15 | 16 | 2 | 1 | 674.534 | 2 | 3059 | 120 | 40.35 | 780.371 | 2 | 3875 | 136 |
| 20 | 30 | 2 | 1 | 33219.9 | 2 | 10625 | 231 | 1117.33 | 35310.7 | 2 | 10625 | 231 |
| 24 | 58 | 2 | 1 | 3929.23 | 2 | 3875 | 136 | 311.94 | $>$14h | 2 | 20475 | 325 |

Experimentations on a 2.4 Ghz Intel core i5 processor based laptop with 3.8 GBytes memory.

# Applications: Best 1-rank tensor approximation

Given a tensor $\mathcal{F} = (\mathcal{F}_{i_1,\ldots,i_m})_{1 \leq i_1 \leq n_1,\ldots,1 \leq i_m \leq n_m} \in \mathbb{R}^{n_1 x n_2 \cdots x n_m}$

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$$\Updownarrow$$

$$\max_{x^1 \in \mathbb{R}^{n_1}, \ldots x^m \in \mathbb{R}^{n_m}} | F(x^1, ..., x^m) |$$

$$s.t \; || x^1 || = \cdots = || x^m || = 1$$

$$where \;\; F(x^1, ..., x^m) = \sum_{1 \le i_1 \le n_1, \ldots, 1 \le i_m \le n_m} \mathcal{F}_{i_1,\ldots,i_m} \cdot (x^1)_{i_1} \cdots (x^m)_{i_m}$$

# Example Best 1-rank symmetric tensor approximation

$\mathcal{F} \in S^3(\mathbb{R}^3)$ with entries
$\mathcal{F}_{111} = 0.0517, \mathcal{F}_{112} = 0.3579, \mathcal{F}_{113} = 0.5298, \mathcal{F}_{122} = 0.7544, \mathcal{F}_{123} = 0.2156,$
$\mathcal{F}_{133} = 0.3612, \mathcal{F}_{222} = 0.3943, \mathcal{F}_{223} = 0.0146, \mathcal{F}_{233} = 0.6718, \mathcal{F}_{333} = 0.9723.$

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$F(x_0, x_1, x_2) = 0.0517x_0^3 + 0.3579x_0^2 x1 + 0.5298x_0^2 x_2 + 0.7544x_0 x_1^2 + 0.2156x_0 x_1 x_2 + 0.3612x_0 x_2^2 + 0.3943x_1^3 + 0.0146x_1^2 x_2 + 0.6718x_1 x_2^2 + 0.9723x_2^3;$

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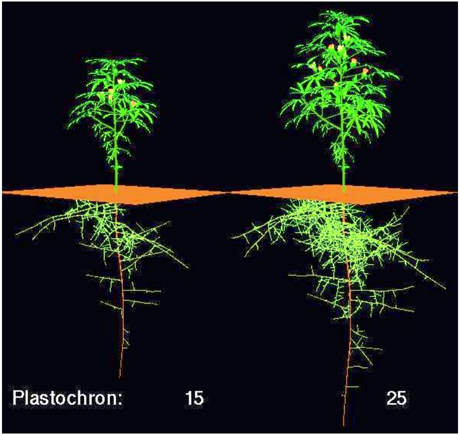
Solution: the rank-1 tensor $\lambda \cdot u^{\otimes 3}$ with $\lambda = 1.093, u = (-0.203893, -0.248661, -0.946887)$
$|| \mathcal{F} - \lambda \cdot u^{\otimes 3} ||^2 = 1.515159281.$

# Applications: Root system model-Stochastic model

- 14 parameters (only 4 estimated on images).
- 15 statistics (size, shape, ...)



# How use our algorithm in this root system model?

- Objective: study which statistics are important.
- Minimize quadratic problem in 15 variables (statistics) where its sum is equal to 1 and all of them are non-negatives
- Solution: The variables which are zero correspond to statistics which are not important.

THANK YOU FOR YOUR ATTENTION!!!