An efficient probabilistic algorithm to compute the real dimension of a real algebraic set.

JNCF 2014

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4th November 2014

Real dimension

Let S be the semi-algebraic set :

$$S = \{x \in \mathbb{R}^n | f_1(x) = \cdots = f_p(x) = 0, g_1(x) > 0, \ldots, g_s(x) > 0\}.$$

with $f_1, \ldots, f_p, g_1, \ldots, g_s$ in $\mathbb{R}[X_1, \ldots, X_n]$

Definition

The real dimension of S is the maximum integer d such that in **generic coordinates**, Interior$(\pi_d(S)) \neq \emptyset$ where $\pi_d : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_d)$.

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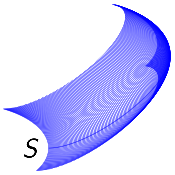

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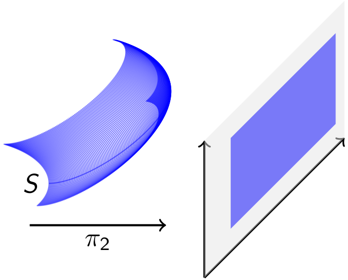

Motivations

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- Computing the set of realizable sign conditions. Barone/Basu 2012
- Computing a bound on the number of connected component of real algebraic sets. Barone/Basu 2013

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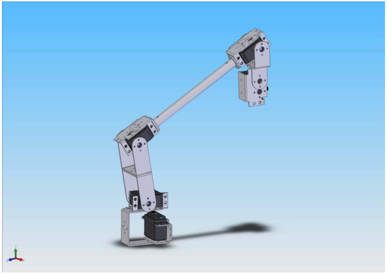
- Applications in mechanics.



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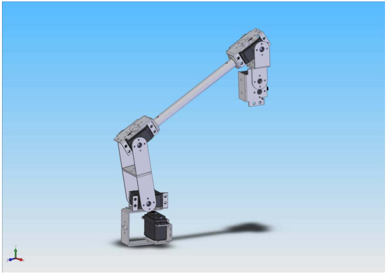

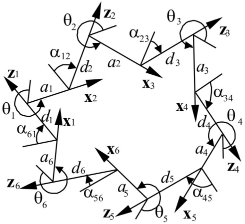

Overconstraint analysis on spatial 6-link loops, Jin/Yang,2002

State-of-the-art

Let $S = \{x \in \mathbb{R}^n | f_1(x) = \cdots = f_p(x) = 0, g_1(x) > 0, \ldots, g_s(x) > 0\}$ with **maximum degree** D and **real dimension** d**.**

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  - **Best implementation** but limited ($n \leq 3$ for non-trivial examples).

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Contribution

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1. New algorithm for **hypersurfaces** $V_{\mathbb{R}}(f)$ (defined by $f = 0$)

General Observation

In the real case,

$$f_1(x) = \cdots = f_p(x) = 0 \Longleftrightarrow f_1^2(x) + \cdots + f_p^2(x) = 0$$

Contribution

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1. New algorithm for **hypersurfaces** $V_{\mathbb{R}}(f)$ (defined by $f = 0$)
2. **Best known** complexity class :

$$\widetilde{O}\left(D^{\mathbf{3d(n-d)}+6n+3}\right)$$

Input :

- f : a polynomial of degree D
- d : the real dimension of $V_{\mathbb{R}}(f)$.

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3. **Probabilistic** algorithm

Probabilistic subroutines

$\rightarrow$ Generic change of variables
$\rightarrow$ One point per connected components and test of emptiness.

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4. **Efficient** implementation

- Checking procedures
- Grobner basis instead of geometric resolution
- Example reached : $n = 6,\ D = 8,$ 130 sec.

Previous approach : Quantifier Elimination (QE)

$$\exists Z \in \mathbb{R}, X^2 + Y^2 + Z^2 - 1 = 0$$

1. Compute Φ quantifier free formula defining $\pi_i(S)$.
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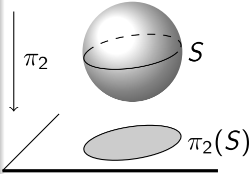

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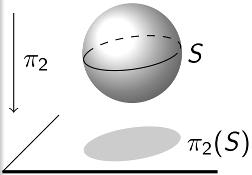

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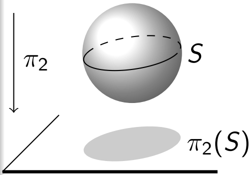

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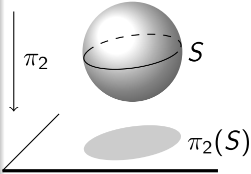

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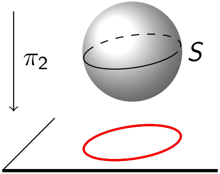

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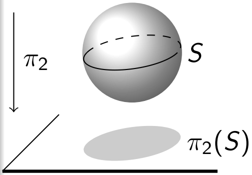

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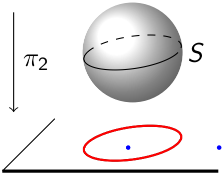

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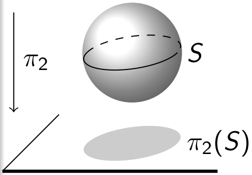

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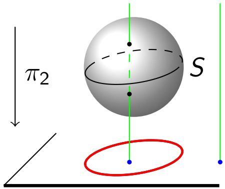

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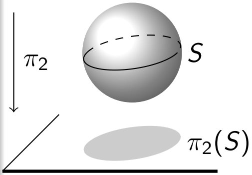

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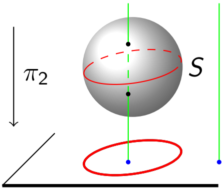

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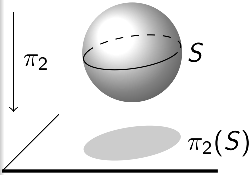

Our appoach : Variant of QE

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   - Deformation + Projection of Polar Varieties.
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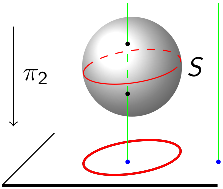


$\rightarrow$ Hypotheses : ~~the algebraic variety assoc. to the polynomial equations is smooth and equidimensional, the projection of S is proper.~~ Hypersurfaces

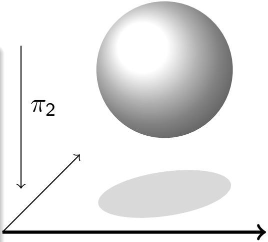
Polar Varieties Todd/Severi (∼30's), Bank/Giusti/Heintz/Mandel/Mbakop

- $V \subset \mathbb{C}^n$ = smooth hypersurface defined by $f = 0$
- $\pi_i : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_i)$

Polar variety W_i associated to V and π_i

$$W_i = \overline{\{x \in V \mid \pi_i(T_x V) \neq \mathbb{C}^i\}}^{\mathrm{Zar}}$$

where $T_x V$ the tangent space to V at x.



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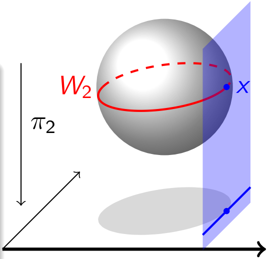

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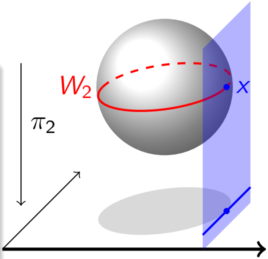

Proposition Hong/Safey (09,12), Safey/Schost (03), Greuet/Safey (13)

If V is compact and smooth, then Boundary$(\pi_i(V)) \subset \pi_i(W_i)$.

Three different cases

$V_{\mathbb{R}}(f)$ a real algebraic set defined by $f = 0$.

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- $\mathrm{Reg}(f) = \mathbb{C}^n - \mathrm{Sing}(f)$

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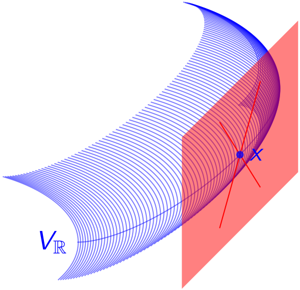

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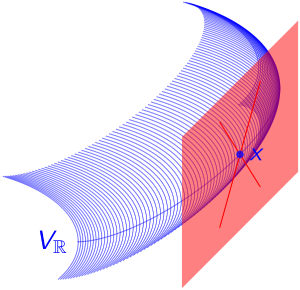

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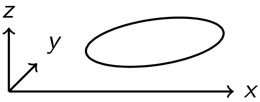
The algorithm on $V_{\mathbb{R}}((x^2+y^2-1)^2+z^2)$

Projection over x :



1. $V_{\mathbb{R}} \subset \mathrm{Sing}(f)$

There is no smooth point

Let x such that $f(x) = f_1(x)^2 + f_2(x)^2 = 0$ then

$$\frac{\partial f}{\partial X_i}(x) = 2f_1(x)\frac{\partial f_1}{\partial X_i}(x) + 2f_2(x)\frac{\partial f_2}{\partial X_i}(x) = 0.$$

The algorithm on $V_{\mathbb{R}}((x^2+y^2-1)^2+z^2)$

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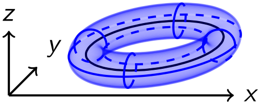

1. $V_{\mathbb{R}} \subset \mathrm{Sing}(f)$
2. Deformation

Infinitesimal deformation and generic coordinates

- V_{ε} defined by $f - \varepsilon = 0$ with ε infinitesimal
    - V_{ε} is smooth
- Generic coordinates $\rightarrow$ random change of variables

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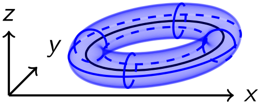

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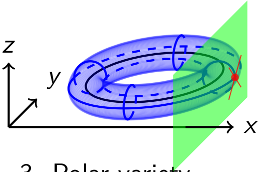


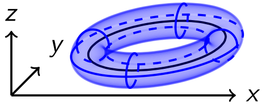
3. Polar variety

Polar varieties

The i-th **polar variety** $W_{\varepsilon,i}$ associated to V_ε and π_i is defined in $\mathbb{C}^n$ by

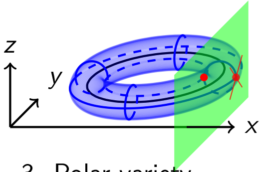
$$f - \varepsilon = \frac{\partial f}{\partial X_{i+1}} = \cdots = \frac{\partial f}{\partial X_n} = 0$$

Projection over x :



1. $V_{\mathbb{R}} \subset \mathrm{Sing}(f)$

2. Deformation



3. Polar variety

Polar varieties

The i-th **polar variety** $W_{\varepsilon,i}$ associated to V_ε and π_i is defined in $\mathbb{C}^n$ by

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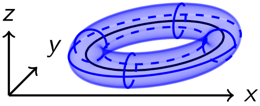

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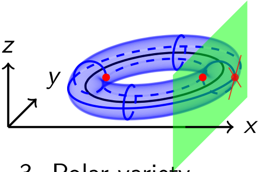

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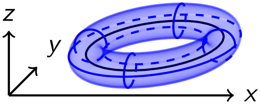

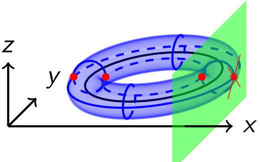

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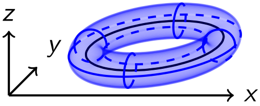

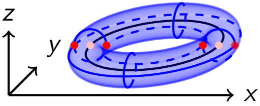

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Main geometric results — B./Safey 2014

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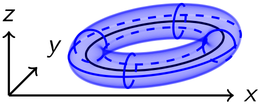

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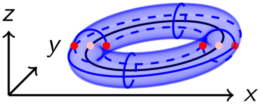

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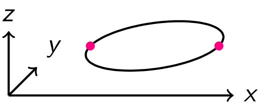
4. Limit



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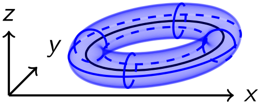

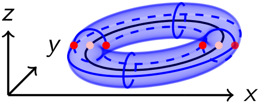

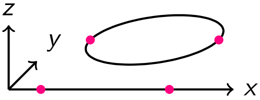

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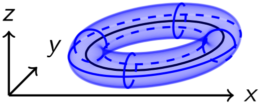

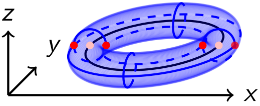

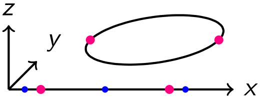

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Main geometric results — B./Safey 2014

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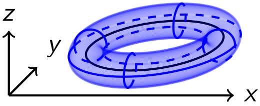

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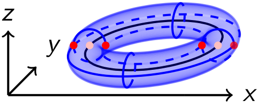

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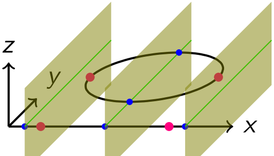

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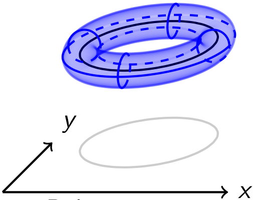
Testing fibers

For each point P in a connected component, **test the emptiness** of

$$\pi_i^{-1}(P) \cap V_{\mathbb{R}}.$$

The algorithm on $V_{\mathbb{R}}((x^2 + y^2 - 1)^2 + z^2)$

Projection over (x, y) :



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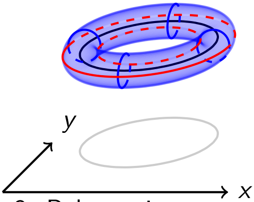

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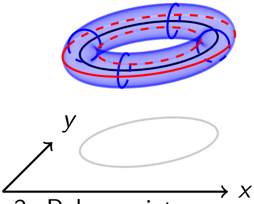

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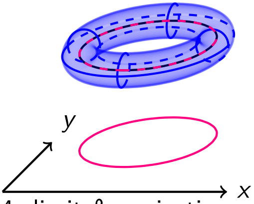

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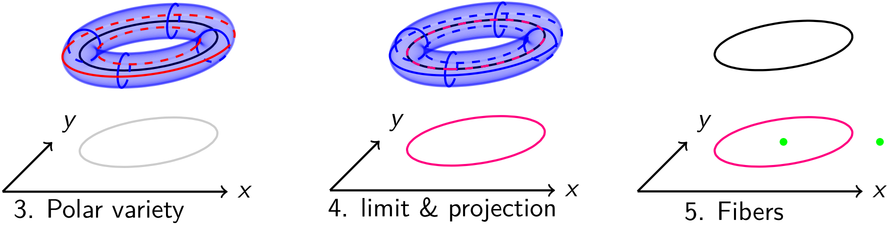

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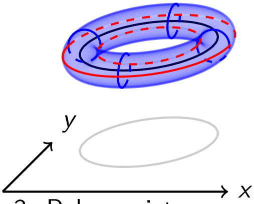

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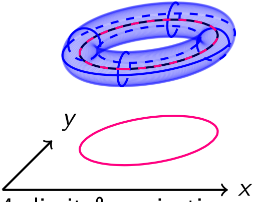

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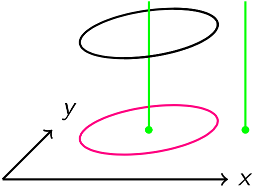

5. Fibers

The interior of the projection is empty so $\dim(V_{\mathbb{R}}) < 2$

Degree

$$W_{\varepsilon,i} = \left\{ x \in \mathbb{C}^n \mid f(x) - \varepsilon = \frac{\partial f}{\partial X_{i+1}}(x) = \cdots = \frac{\partial f}{\partial X_n}(x) = 0 \right\}$$

- $\deg(\lim_{\varepsilon \to 0} W_{\varepsilon,i}) \leq D^{n-i}$
- $\deg(\pi_i(\lim_{\varepsilon \to 0} W_{\varepsilon,i})) \leq D^{n-i}$

Complexity

- Computing $\pi_i(\lim_{\varepsilon \to 0} W_{\varepsilon,i})$: Lecerf 01, Schost 03

$$\widetilde{O}\left(D^{(n-i)i+4n+8}\right)$$

- Computing one point per connected component of $\mathbb{R}^i - \pi_i(\lim_{\varepsilon \to 0} W_{\varepsilon,i})$: Safey/Schost 03

$$\widetilde{O}\left(D^{3(n-i)i+6n}\right)$$

- Complexity algorithm :

$$\widetilde{O}\left(D^{3(n-d)d+6n}\right)$$

Computational tools

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- **Gröbner basis elimination** to compute polynomials defining $\pi_i(\lim_{\varepsilon \to 0} W_{\varepsilon,i}) \supset \text{Boundary}(\pi_i(V_{\mathbb{R}}))$.
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- Computing **One Point per Connected Components** and testing **emptiness** of a semi-algebraic set.
    - **RAG** library in Maple, by M. Safey El Din.

Benchmark : random polynomials in n variables

- CAD : multithread implementation in Maple.

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∞ means $> 24\,hours$

n	s	degrees	d	CAD	DIM	
4	1	2	3	0.3 sec.	6 sec.	
4	2	2,2	2	∞	27 sec.	
4	3	2,1,1	1	∞	58 sec.	
5	1	2	4	3 sec.	7 sec.	
5	2	2,2	3	∞	2324 sec.	
5	3	2,2,1	2	∞	388 sec.	
5	4	2,1,1,1	1	∞	141 sec.	
6	1	2	5	2.2 sec.	9 sec.	
6	2	2,2	4	∞	185 sec.	
6	3	2,1,1	3	∞	1253 sec.	
6	4	2,1,1,1	2	∞	11 hours.	
6	5	2,1,1,1,1	1	∞	325 sec.	

Table : Running time for $V_{\mathbb{R}}(f_1^2 + \cdots + f_s^2)$, random polynomials in n variables

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- Step 1 : computing $\pi_i(\lim_{\varepsilon \to 0} W_{\varepsilon,i})$, Step 2 : one point per connected components, Step 3 : testing fibers.

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n	s	degrees	d	CAD	DIM	Step 1	Step 2	Step 3	# fibers
4	1	2	3	0.3 sec.	6 sec.	49%	1%	50%	1
4	2	2,2	2	∞	27 sec.	5%	75%	20%	49
4	3	2,1,1	1	∞	58 sec.	3%	6%	91%	38
5	1	2	4	3 sec.	7 sec.	/	/	/	
5	2	2,2	3	∞	2324 sec.	0%	95%	5%	77
5	3	2,2,1	2	∞	388 sec.	2%	55%	43%	
5	4	2,1,1,1	1	∞	141 sec.	1%	27%	72%	46
6	1	2	5	2.2 sec.	9 sec.	/	/	/	
6	2	2,2	4	∞	185 sec.	0.3%	10.7%	89%	132
6	3	2,1,1	3	∞	1253 sec.	0.3%	95.7%	4%	17
6	4	2,1,1,1	2	∞	11 hours.	0%	99.6%	0.4%	
6	5	2,1,1,1,1	1	∞	325 sec.	1%	21%	78%	25

Table : Running time for $V_{\mathbb{R}}(f_1^2 + \cdots + f_s^2)$, random polynomials in n variables

Polynomials naturally sum of square of polynomials

- Input : discriminant of the characteristic polynomial of a linear matrix.

n	k	D	d	CAD	DIM	Step 1	Step 2	Step 3	#fibers
3	2	2	1	0.02 sec.	1.2 sec.	71%	7%	22%	4
3	3	6	1	220 sec.	4 sec.	72%	7%	21%	3
3	4	12	1	∞	248 sec.	1%	0%	99%	2
4	2	2	2	0.06 sec.	2 sec	57%	10%	33%	4
4	3	6	2	∞	16 sec.	5%	46%	49%	77
5	2	2	3	0.08 sec.	2 sec.	53%	10%	37%	4
5	3	6	3	∞	213 sec.	1%	92%	7%	123
6	2	2	4	0.06 sec.	3 sec.	39%	13%	48%	4
6	3	6	4	∞	600 sec.	9%	64%	27%	133
7	2	2	5	0.12 sec.	1.8 sec.	70%	10%	20%	4

Table : Running time for $V_{\mathbb{R}}(f)$. Linear matrix of size k.

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4	2	2	2	0.06 sec.	2 sec	57%	10%	33%	4
4	3	6	2	∞	16 sec.	5%	46%	49%	77
5	2	2	3	0.08 sec.	2 sec.	53%	10%	37%	4
5	3	6	3	∞	213 sec.	1%	92%	7%	123
6	2	2	4	0.06 sec.	3 sec.	39%	13%	48%	4
6	3	6	4	∞	600 sec.	9%	64%	27%	133
7	2	2	5	0.12 sec.	1.8 sec.	70%	10%	20%	4

Table : Running time for $V_{\mathbb{R}}(f)$. Linear matrix of size k.

- Polynomial Voronoi I : $n = 6$, deg $= 8$: $dim = 4$ with 130 sec.

Everet/Lazard/Lazard/Safey 09

Serie I : $f_n := (\sum_{i=1}^n x_i^2)^2 - \sum_{i=1}^{n-1} x_i^2 x_{i+1}^2 - x_n^2 x_1^2$
Serie II : $f_n := \sum_{i=1}^n x_i^6 - 3\prod_{i=1}^n x_i^2$
Serie III : $f_n := \prod_{i=1}^n x_i^2 + n - 1 - n^{n-2}(\sum_{i=1}^n x_i)^2$

Serie	n	D	d	CAD	DIM	Step 1	Step 2	Step 3	#fibers
I	3	4	2	0.9 sec.	1.5 sec.	85%	7%	8%	4
	4	4	3	0.3 sec.	3.4 sec.	62%	1%	37%	1
	5	4	3	17 sec.	34 sec.	1%	88%	11%	144
	6	4	4	5500 sec.	95 sec.	0.4%	90%	9.6%	256
	7	4	5	∞	300 sec.	0.1%	93%	6.9%	384
II	3	6	1	1.06sec.	3.865 sec.	80%	7%	13%	3
	5	6	4	∞	8.5 hours	0%	100%	0%	163
III	3	4	0	1.74 sec.	5.5 sec.	31%	0.7%	68.3%	3
	4	4	0	40 sec.	68 sec.	4%	0.1%	95.9%	7
	5	4	0	∞	7680 sec.	10%	0%	90%	3

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Perspectives

Future work

- Generalization to real algebraic and semi-algebraic sets

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Motivation for connectivity queries of curves in $\mathbb{R}^n$

- Experimentally : too many fibers
- Goal : only one point per connected components
- Idea : connectivity queries of curves in $\mathbb{R}^n$ and roadmap Safey/Schost 2014