Structure of polyzetas and the algorithms to express them on algebraic bases on words

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Outline

- Introduction
- Hopf algebras of noncommutative polynomials
- Global regularizations of polyzetas
- Polynomial relations among polyzetas

- **Introduction**
- Hopf algebras of noncommutative polynomials
- Global regularizations of polyzetas
- Polynomial relations among polyzetas

Definition

For each $s = (s_1, \ldots, s_r) \in (\mathbb{N}^*)^r, s_1 > 1$, the polyzetas (multiple zeta values (MZVs)) is defined by the following convergent series

$$\zeta(s) = \zeta(s_1, \ldots, s_r) := \sum_{n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}$$

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Example

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3},$$

$$\zeta(2,3) = \sum_{n_1 > n_2 > 0} \frac{1}{n_1^2 n_2^3}.$$

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Theorem (comparison formula) [9]

$$\begin{aligned} Z_\gamma &= \Gamma(y_1 + 1)\pi_Y(Z_{⧢}) & (1) \\ \iff Z_{⧺} &= B'(y_1)\pi_Y(Z_{⧢}) & (2) \end{aligned}$$

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- **Hopf algebras of noncommutative polynomials**
- Global regularizations of polyzetas
- Polynomial relations among polyzetas

On the alphabet $X = \{x_0, x_1\}$ [4]

Hopf algebras in duality

$$(\mathbb{Q}\langle X\rangle, ., 1_{X^*}, \Delta_{\sqcup\!\sqcup}, s) \rightleftarrows (\mathbb{Q}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*}, \Delta_{conc}, s),$$

One constructed the PBW basis $(P_w)_{w\in X^*}$ of the freely associated algebra $\mathbb{Q}\langle X\rangle$ and the transcendent basis $(S_l)_{l\in\mathcal{L}ynX}$ of the algebra $(\mathbb{Q}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*})$.

Algebraic structures on noncommutative polynomial

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On the alphabet $Y = (y_s)_{s\in\mathbb{N}^*}$ [2]

Hopf algebras in duality

$$(\mathbb{Q}\langle Y\rangle, ., 1_{Y^*}, \Delta_{\uplus}, s') \rightleftarrows (\mathbb{Q}\langle Y\rangle, \uplus, 1_{Y^*}, \Delta_{conc}, s'),$$

We also constructed the PBW basis $(\Pi_w)_{w\in Y^*}$ of the freely associated algebra $\mathbb{Q}\langle Y\rangle$ and the transcendent basis $(\Sigma_l)_{l\in\mathcal{L}ynY}$ of the algebra $(\mathbb{Q}\langle Y\rangle, \uplus, 1_{Y^*})$.

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We now consider homogeneous polynomials by length with respect to alphabet X and by weight with respect to alphabet Y. We view the following graded vector spaces

$$\mathcal{C} := \mathbb{Q} \oplus \mathbb{Q}\langle X\rangle x_1 = \bigoplus_{n\geq 0} span_{\mathbb{Q}}(X_n) \simeq \mathbb{Q}\langle Y\rangle = \bigoplus_{n\geq 0} span_{\mathbb{Q}}(Y_n)$$

where X_n, Y_n being corresponding the sets of words with length and weight n.

Example

$$\begin{aligned}
X_0 &:= \{1\}, & Y_0 &:= \{1\}\\
X_1 &:= \{x_1\}, & Y_1 &:= \{y_1\}\\
X_2 &:= \{x_0x_1, x_1x_1\}, & Y_2 &:= \{y_2, y_1^2\}\\
X_3 &:= \{x_0^2x_1, x_0x_1^2, x_1x_0x_1, x_1^3\}, & Y_3 &:= \{y_3, y_2y_1, y_1y_2, y_1^3\}\\
&\ldots
\end{aligned}$$

Constructing linear relation between $\mathcal{C} := \mathbb{Q} \oplus \mathbb{Q}\langle X\rangle x_1$ and $\mathbb{Q}\langle Y\rangle$, view as the graded modules (2/4)

Definition

Let us define the linear isomorphism

$$\begin{array}{rcl} \pi_Y : \mathcal{C} & \longrightarrow & \mathbb{Q}\langle Y\rangle \\ x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1 & \longmapsto & y_{s_1} \ldots y_{s_r} \end{array}$$

and π_X be its inverse.

Its extension over $\mathbb{Q}\langle X\rangle$, be still denoted by π_Y, satisfying $\pi_Y(p) = 0$ for any $p \in \mathbb{Q}\langle X\rangle x_0$.

Example

$$\pi_Y(x_0) = 0, \pi_Y(x_1) = y_1$$
$$\pi_Y(x_0x_1) = y_2$$
$$\pi_Y(2x_0x_1 - x_1x_0 - \frac{1}{2}x_1x_0x_1) = 2y_2 - \frac{1}{2}y_1y_2$$

Constructing linear relation between $\mathcal{C} := \mathbb{Q} \oplus \mathbb{Q}\langle X\rangle x_1$ and $\mathbb{Q}\langle Y\rangle$, view as the graded modules (3/4)

Lemma

For any $w \in X^* x_1$, we call P'_w, S'_w to be corresponding P_w, S_w restricted on $\mathcal{C}$. Then, $(P'_w)_{w \in X_n}, (S'_w)_{w \in X_n}$ are a pair of bases in duality of the $\mathrm{span}_{\mathbb{Q}}(X_n)$.

Example

$$\begin{array}{llll}
P_{x_1} = S_{x_1} = x_1 & \in \mathcal{C} & \Rightarrow P'_{x_1} & = S'_{x_1} = x_1, \\
P_{x_0x_1} = x_0x_1 - x_1x_0 & \notin \mathcal{C} & \Rightarrow P'_{x_0x_1} & = x_0x_1, \\
S_{x_0x_1} = x_0x_1 & \in \mathcal{C} & \Rightarrow S'_{x_0x_1} & = x_0x_1, \\
P_{x_0x_1^2} = x_0x_1^2 - 2x_1x_0x_1 + x_1^2x_0 & \notin \mathcal{C} & \Rightarrow P'_{x_0x_1^2} & = x_0x_1^2 - 2x_1x_0x_1, \\
S_{x_0x_1x_1} = x_0x_1^2 & \in \mathcal{C} & \Rightarrow S'_{x_0x_1} & = x_0x_1
\end{array}$$

Remark: For any $w \in X^*$, we have $\pi_Y(P'_w) = \pi_Y(P_w)$ and $\pi_Y(S'_w) = \pi_Y(S_w)$.

Constructing linear relation between $\mathcal{C} := \mathbb{Q} \oplus \mathbb{Q}\langle X \rangle x_1$ and $\mathbb{Q}\langle Y \rangle$, view as the graded modules $(4/4)$

For each n, we arrange the elements of X_n and Y_n in the increasing order respectively as $u_1^{(n)} < u_2^{(n)} < \ldots < u_{2^{n-1}}^{(n)}$, $v_1^{(n)} < v_2^{(n)} < \ldots < v_{2^{n-1}}^{(n)}$; and then establishing the matrix representation of π_Y in the two ordered bases as follow

$$\begin{pmatrix} \pi_Y P'_{u_1^{(n)}} \\ \vdots \\ \pi_Y P'_{u_{2^{n-1}}^{(n)}} \end{pmatrix} = M^{(n)} \begin{pmatrix} \Pi_{v_1^{(n)}} \\ \vdots \\ \Pi_{v_{2^{n-1}}^{(n)}} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \pi_Y S'_{u_1^{(n)}} \\ \vdots \\ \pi_Y S'_{u_{2^{n-1}}^{(n)}} \end{pmatrix} = N^{(n)} \begin{pmatrix} \Sigma_{v_1^{(n)}} \\ \vdots \\ \Sigma_{v_{2^{n-1}}^{(n)}} \end{pmatrix}.$$

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By the duality, we have $N^{(n)} := ({}^t(M^{(n)}))^{-1}$. We can see more clearly by the following diagram

$$\begin{array}{ccc}
(span_{\mathbb{Q}}(X_n), (P'_{u_i^{(n)}})_{1\le i\le 2^{n-1}}) & \xrightarrow[M^{(n)}]{\pi_Y} & (span_{\mathbb{Q}}(Y_n), (\Pi_{v_j^{(n)}})_{1\le j\le 2^{n-1}}) \\
\Big\downarrow \text{duality} & & \Big\downarrow \text{duality} \\
(span_{\mathbb{Q}}(X_n), (S'_{u_i^{(n)}})_{1\le i\le 2^{n-1}}) & \xrightarrow[N^{(n)}]{\pi_Y} & (span_{\mathbb{Q}}(Y_n), (\Sigma_{v_j^{(n)}})_{1\le j\le 2^{n-1}})
\end{array}$$

Encoding polyzetas by words

Definition

One defines the two morphisms $\zeta_{\sqcup\!\sqcup} : (\mathcal{C}, \sqcup\!\sqcup) \to (\mathbb{R}, \cdot)$ and $\zeta_{\uplus} : (\mathbb{Q}\langle Y\rangle, \uplus) \to (\mathbb{R}, \cdot)$ map each word (called convergent word) form $x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1 \in x_0X^*x_1$ or $y_{s_1} \ldots y_{s_r} \in Y^* \setminus y_1 Y^*$ become $\zeta(s_1, \ldots, s_r)$ and convention $\zeta_{\sqcup\!\sqcup}(x_1) = \zeta_{\uplus}(y_1) = 0$.

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The global regularization [7, 9]

$$
\begin{aligned}
Z_{\uplus} &= B'(y_1)\pi_Y Z_{\sqcup\!\sqcup} \\
\Leftrightarrow \sum_{w\in Y^*} \zeta_{\uplus}(\Sigma_w)\Pi_w &= \exp\left(\sum_{k\geq 2} \frac{(-1)^{k-1}\zeta(k)}{k} y_1^k\right) \sum_{w\in X^*} \zeta_{\sqcup\!\sqcup}(S'_w)\pi_Y P'_w
\end{aligned}
$$

Lemma

If we have the representation $B'(y_1) = 1 + \sum_{m\geq 2} B^{(m)} y_1^m$ *then*

$$B^{(m)} = \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} \sum_{\substack{k_1,\ldots,k_i\geq 2\\ k_1+\ldots+k_i=m}} (-1)^{m-i} \frac{\zeta(k_1)\ldots\zeta(k_i)}{k_1\ldots k_i}, \qquad (3)$$

where $\lfloor \frac{m}{2} \rfloor$ *is the largest integer not greater than* $\frac{m}{2}$.

Example

$$\begin{aligned}
B^{(2)} &= -\frac{1}{2}\zeta(2) \\
B^{(3)} &= \frac{1}{3}\zeta(3) \\
B^{(4)} &= -\frac{1}{4}\zeta(4) + \frac{1}{2^2}\zeta(2)^2 \\
B^{(5)} &= \frac{1}{5}\zeta(5) - 2\frac{\zeta(2)}{2}\frac{\zeta(3)}{3}
\end{aligned}$$

Identify the coefficients $(1/3)$

on basis $(\Pi_w)_{w\in Y^*}$

$$\sum_{v\in Y^*} \zeta_{⧢}(\Sigma_v)\Pi_v = B'(y_1) \sum_{u\in X^*} \zeta_{\sqcup\!\sqcup}(S_u)\pi_Y P_u$$

$$\Leftrightarrow \sum_{n\geq 1}\sum_{j=1}^{2^{n-1}} \zeta_{⧢}(\Sigma_{v_j^{(n)}})\Pi_{v_j^{(n)}} = B'(y_1) \sum_{n\geq 1}\sum_{j=1}^{2^{n-1}} \zeta_{\sqcup\!\sqcup}(\pi_X(\Sigma_{v_j^{(n)}}))\Pi_{v_j^{(n)}}$$

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on basis $(P'_w)_{w\in X^*x_1}$

$$\pi_X\left(\sum_{v\in Y^*}\zeta_{\uplus}(\Sigma_v)\Pi_v\right) = \pi_X\left(B'(y_1)\sum_{u\in X^*}\zeta_{\sqcup\sqcup}(S_u)\pi_Y P_u\right)$$

$$\Leftrightarrow \sum_{n\geq 1}\sum_{i=1}^{2^{n-1}}\zeta_{\uplus}(\pi_Y(S'_{u_i^{(n)}}))P'_{u_i^{(n)}} = B'(x_1)\sum_{n\geq 1}\sum_{i=1}^{2^{n-1}}\zeta_{\sqcup\sqcup}(S'_{u_i^{(n)}})P'_{u_i^{(n)}}$$

Identify the coefficients (2/3)

on basis $(\Pi_w)_{w\in Y^*}$

$$\zeta_{\uplus}(\Sigma_{v_j^{(n)}}) = \sum_{i=1}^{2^{n-1}} M_{ij}^{(n)} \zeta_{\sqcup\sqcup}(S_{u_i^{(n)}}), \quad \forall v_j^{(n)} \notin y_1^2 Y^*$$

$$\zeta_{\uplus}(\Sigma_{v_j^{(n)}}) = \zeta_{\sqcup\sqcup}(\pi_Y(\Sigma_{v_j^{(n)}})) + \sum_{m=2}^{k} B^{(m)} \zeta_{\sqcup\sqcup}(\pi_X(\Sigma_{y_1^{k-m}w})), \ \forall v_j^{(n)} = y_1^k w$$

Example

$$\begin{aligned}
\zeta(\Sigma_{y_2}) &= \zeta(S_{x_0x_1}) \\
\zeta(\Sigma_{y_1^2}) &= \left(\frac{1}{2}\zeta(S_{x_0x_1}) + \zeta(S_{x_1^2})\right) + \left(-\frac{1}{2}\zeta(2)\right) \\
\zeta(\Sigma_{y_1^3}) &= \left(\frac{1}{6}\zeta(S_{x_0^2x_1}) - \frac{1}{2}\zeta(S_{x_0x_1^2}) + \frac{1}{2}\zeta(S_{x_1x_0x_1}) + \zeta(S_{x_1^3})\right) \\
&\quad + \left(-\frac{1}{2}\zeta(2)\zeta(S_{x_1}) + \frac{1}{3}\zeta(3)\right)
\end{aligned}$$

Identify the coefficients (3/3)

on basis $(P'_w)_{w \in X^* x_1}$

$$\sum_{j=1}^{2^{n-1}} N_{ij}^{(n)} \zeta_{⧢}(\Sigma_{v_j^{(n)}}) = \zeta_{ш}(S_{u_i^{(n)}}), \quad \forall u_i^{(n)} \notin x_1^2 X^* x_1$$

$$\sum_{j=1}^{2^{n-1}} N_{ij}^{(n)} \zeta_{⧢}(\Sigma_{v_j^{(n)}}) = \zeta_{ш}(S_{u_i^{(n)}}) + \sum_{m=2}^{k} B^{(m)} \zeta_{ш}((S_{x_1^{k-m} w})), \ \forall u_i^{(n)} = x_1^k w$$

Example

$$\begin{aligned}
\zeta_{⧢}(\Sigma_{y_2}) &= \zeta_{ш}(S_{x_0 x_1}) \\
\zeta_{⧢}(\Sigma_{y_1^2}) - \frac{1}{2}\zeta_{⧢}(\Sigma_{y_2}) &= \zeta(S_{x_0 x_1}) + \left(-\frac{1}{2}\zeta(2)\right) \\
\zeta_{⧢}(\Sigma_{y_1^3}) - \frac{1}{2}\zeta_{⧢}(\Sigma_{y_1 y_2}) + \frac{1}{3}\zeta_{⧢}(\Sigma_{y_3}) &= \zeta_{ш}(S_{x_1^3}) \\
&\quad + \left(-\frac{\zeta(2)}{2}\zeta_{ш}(S_{x_1}) + \frac{\zeta(3)}{3}\right)
\end{aligned}$$

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- **Polynomial relations among polyzetas**

INPUT: One positive integer N.

1. Find all words, w, of length n in X^* such that $w \in x_0X^*x_1$ or $w \in x_1x_0X^*x_1$. We call this set to be $X^{(n)}$.
2. Find all Lyndon words of weight n in Y^*.
3. Find relations of polyzetas of weight n expressed follow $(\Sigma_l)_{l\in\mathcal{L}ynY}$ by the way: for each $w \in X^{(n)}$, take $\mathcal{P} := \pi_Y(S_w)$ and express $\mathcal{P}$ by basis $(\Sigma_l)_{l\in\mathcal{L}ynY}$ (use the methode in [1]), called $\mathcal{P}_\Sigma$
   - **i)** if $w \in \mathcal{L}ynX$ then we store $\zeta(\mathcal{P}_\Sigma)$ to variable $\zeta(S_w)$,
   - **ii)** if $w = x_1u$, $u \in x_0X^*x_1$ then we take $\zeta(\mathcal{P}_\Sigma)$ and set a relation $\zeta(\mathcal{P}_\Sigma) = 0$ (because $\zeta(S_w) = \zeta(x_1 ⧢ S_u) = 0$).
   - **iii)** if $w \in x_0X^*x_1 \setminus \mathcal{L}ynX$, we rewrite w form Lyndon factorization $w = l_1^{i_1} \ldots l_k^{i_k}$, get $\zeta(S_{l_j})$, $j = 1..k$ from data of lower weight and establish the relation

$$\frac{1}{i_1! \ldots i_k!}\zeta(S_{l_1})^{i_1} \ldots \zeta(S_{l_k})^{i_k} = \zeta(\mathcal{P}_\Sigma)$$

4. Eliminate the system of relations to find down structure of polyzetas of weight n on $(\Sigma_l)_{l\in\mathcal{L}ynY}$.

OUTPUT: Polynomial relations among polyzetas up to weight N.

Polynomial relations among polyzetas

3	$\zeta(\Sigma_{y_2y_1})$	$= \frac{3}{2}\zeta(\Sigma_{y_3})$	$\zeta(S_{x_0x_1^2})$	$= \zeta(S_{x_0^2x_1})$
4	$\zeta(\Sigma_{y_4})$	$= \frac{2}{5}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^3x_1})$	$= \frac{2}{5}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_3y_1})$	$= \frac{3}{10}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^2x_1^2})$	$= \frac{1}{10}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_2y_1^2})$	$= \frac{2}{3}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0x_1^3})$	$= \frac{2}{5}\zeta(S_{x_0x_1})^2$
5	$\zeta(\Sigma_{y_3y_2})$	$= 3\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - 5\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^3x_1^2})$	$= -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_4y_1})$	$= -\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{2}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1x_0x_1})$	$= -\frac{3}{2}\zeta(S_{x_0^4x_1}) + \zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})$
	$\zeta(\Sigma_{y_2^2y_1})$	$= \frac{3}{2}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - \frac{25}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1^3})$	$= -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_3y_1^2})$	$= \frac{5}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1x_0x_1^2})$	$= \frac{1}{2}\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_2y_1^3})$	$= \frac{1}{4}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{4}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1^4})$	$= \zeta(S_{x_0^4x_1})$
6	$\zeta(\Sigma_{y_6})$	$= \frac{8}{35}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^5x_1})$	$= \frac{8}{35}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_2})$	$= \zeta(\Sigma_{y_3})^2 - \frac{4}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^4x_1^2})$	$= \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_5y_1})$	$= \frac{2}{7}\zeta(\Sigma_{y_2})^3 - \frac{1}{2}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1x_0x_1})$	$= \frac{4}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_3y_1y_2})$	$= -\frac{17}{30}\zeta(\Sigma_{y_2})^3 + \frac{9}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1^3})$	$= \frac{23}{70}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_2y_1})$	$= 3\zeta(\Sigma_{y_3})^2 - \frac{9}{10}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^2x_1x_0x_1^2})$	$= \frac{2}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_1^2})$	$= \frac{3}{10}\zeta(\Sigma_{y_2})^3 - \frac{3}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^2x_0x_1})$	$= -\frac{89}{210}\zeta(S_{x_0x_1})^3 + \frac{3}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2^2y_1^2})$	$= \frac{11}{63}\zeta(\Sigma_{y_2})^3 - \frac{1}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^4})$	$= \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_1^3})$	$= \frac{1}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0x_1x_0x_1^3})$	$= \frac{8}{21}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2y_1^4})$	$= \frac{17}{50}\zeta(\Sigma_{y_2})^3 + \frac{3}{16}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0x_1^5})$	$= \frac{8}{35}\zeta(S_{x_0x_1})^3$

List of the irreducible polyzetas up to weight 12:

$\zeta(\Sigma_{y_2}), \zeta(\Sigma_{y_3}), \zeta(\Sigma_{y_5}), \zeta(\Sigma_{y_7})$	$\zeta(S_{x_0x_1}), \zeta(S_{x_0^2x_1}), \zeta(S_{x_0^4x_1}), \zeta(S_{x_0^6x_1}),$
$\zeta(\Sigma_{y_3y_1^5}), \zeta(\Sigma_{y_9}), \zeta(\Sigma_{y_3y_1^7}), \zeta(\Sigma_{y_{11}})$	$\zeta(S_{x_0x_1^2x_0x_1^4}), \zeta(S_{x_0^8x_1}), \zeta(S_{x_0x_1^2x_0x_1^6}), \zeta(S_{x_0^{10}x_1})$
$\zeta(\Sigma_{y_2y_1^9}), \zeta(\Sigma_{y_3y_1^9}), \zeta(\Sigma_{y_2^2y_1^8})$	$\zeta(S_{x_0x_1^3x_0x_1^7}), \zeta(S_{x_0x_1^2x_0x_1^8}), \zeta(S_{x_0x_1^4x_0x_1^6})$

List of the irreducible polyzetas up to weight 12:

$\zeta(\Sigma_{y_2}), \zeta(\Sigma_{y_3}), \zeta(\Sigma_{y_5}), \zeta(\Sigma_{y_7})$	$\zeta(S_{x_0x_1}), \zeta(S_{x_0^2x_1}), \zeta(S_{x_0^4x_1}), \zeta(S_{x_0^6x_1}),$
$\zeta(\Sigma_{y_3y_1^5}), \zeta(\Sigma_{y_9}), \zeta(\Sigma_{y_3y_1^7}), \zeta(\Sigma_{y_{11}})$	$\zeta(S_{x_0x_1^2x_0x_1^4}), \zeta(S_{x_0^8x_1}), \zeta(S_{x_0x_1^2x_0x_1^6}), \zeta(S_{x_0^{10}x_1})$
$\zeta(\Sigma_{y_2y_1^9}), \zeta(\Sigma_{y_3y_1^9}), \zeta(\Sigma_{y_2^2y_1^8})$	$\zeta(S_{x_0x_1^3x_0x_1^7}), \zeta(S_{x_0x_1^2x_0x_1^8}), \zeta(S_{x_0x_1^4x_0x_1^6})$

Let us now denote Z_n to be the vector space generated by polyzetas of weight n. The result also verify the Zagier's dimension conjecture.

- $n = 2, d_2 = 1, Z_2 = span\{\zeta(\Sigma_{y_2})\} = span\{\zeta(Sx_0x_1)\}$
- $n = 3, d_3 = 1, Z_3 = span\{\zeta(\Sigma_{y_3})\} = span\{\zeta(S_{x_0^2x_1})\}$
- $n = 4, d_4 = 1, Z_4 = span\{\zeta(\Sigma_{y_2})^2\} = span\{\zeta(S_{x_0x_1})^2\}$
- $n = 5, d_5 = 2, Z_5 = span\{\zeta(\Sigma_{y_5}), \zeta(\Sigma_{y_2})\zeta(\Sigma_{y_3})\} = span\{\zeta(S_{x_0^4x_1}), \zeta(S_{x_0x_1})\zeta(S_{x_0^2x_1})\}$
- $n = 6, d_6 = 2, Z_6 = span\{\zeta(\Sigma_{y_2})^3, \zeta(\Sigma_{y_3})^2\} = span\{\zeta(S_{x_0x_1})^3, \zeta(S_{x_0^2x_1})^2\}$
- $n = 7, d_7 = 3, Z_7 = span\{\zeta(\Sigma_{y_7}), \zeta(\Sigma_{y_2})\zeta(\Sigma_{y_5}), \zeta(\Sigma_{y_2})^2\zeta(\Sigma_{y_3})\} =$ $span\{\zeta(S_{x_0^6x_1}), \zeta(S_{x_0x_1})\zeta(S_{x_0^4x_1}), \zeta(S_{x_0x_1})^2\zeta(S_{x_0^2x_1})\}$
- $n = 8, d_8 = 4, Z_8 = span\{\zeta(\Sigma_{y_2})^4, \zeta(\Sigma_{y_3})\zeta(\Sigma_{y_5}), \zeta(\Sigma_{y_2})\zeta(\Sigma_{y_3})^2, \zeta(\Sigma_{y_3y_1^5})\} =$ $span\{\zeta(S_{x_0x_1})^4, \zeta(S_{x_0^2x_1})\zeta(S_{x_0^4x_1}), \zeta(S_{x_0x_1})\zeta(S_{x_0^2x_1})^2, \zeta(S_{x_0x_1^2x_0x_1^4})\}$
- $n = 9, d_9 = 5, Z_9 = span\{\zeta(\Sigma_{y_9}), \zeta(\Sigma_{y_2})^2\zeta(\Sigma_{y_5}), \zeta(\Sigma_{y_2})\zeta(\Sigma_{y_7}), \zeta(\Sigma_{y_2})^3\zeta(\Sigma_{y_3}), \zeta(\Sigma_{y_3})^3\} =$ $span\{\zeta(S_{x_0^8x_1}), \zeta(S_{x_0x_1})^2\zeta(S_{x_0^4x_1}), \zeta(S_{x_0x_1})\zeta(S_{x_0^6x_1}), \zeta(S_{x_0x_1})^3\zeta(S_{x_0^2x_1}), \zeta(S_{x_0^2x_1})^3\}$
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THANK YOU FOR YOUR ATTENTION