Computations of Hypergeometric Generating Series for Small-Step Walks in the Quarter Plane

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Joint work with A. Bostan, M. Kauers, L. Pech, and M. van Hoeij

▷ Nearest-neighbor walks in the quarter plane = walks in $\mathbb{N}^2$ starting at $(0,0)$ and using steps in a prefixed subset $\mathfrak{S}$ of

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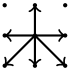
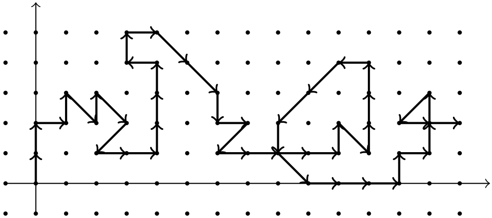
▷ Example with $n = 45$, $i = 14$, $j = 2$ for:

$\mathfrak{S} =$ 



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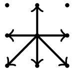
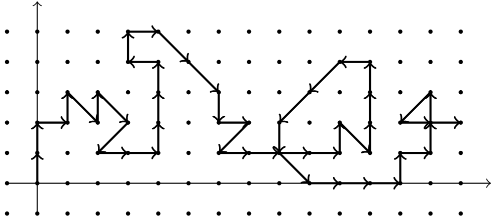
$\mathfrak{S} =$ 



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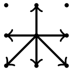
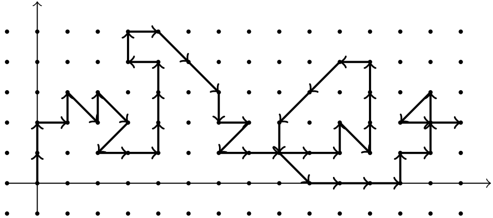
$\mathfrak{S} =$ 



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▷ Special, combinatorially meaningful specializations:

- $f_{n;0,0}$ counts walks returning to the origin, a.k.a. excursions;
- $f_n = \sum_{i,j\geq 0} f_{n;i,j}$ counts walks with prescribed length.

Generating Series and Combinatorial Problems

▷ Complete generating series:

$$F(t;x,y) = \sum_{n=0}^{\infty} \left(\sum_{i,j=0}^{\infty} f_{n;i,j} x^i y^j \right) t^n \ \in \mathbb{Q}[x,y][[t]].$$

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Combinatorial questions: Given $\mathfrak{S}$, what can be said about $F(t;x,y)$, resp. $f_{n;i,j}$, and their variants?

- Algebraic nature of F: algebraic? transcendental?
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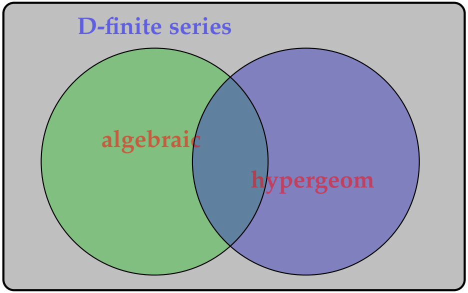
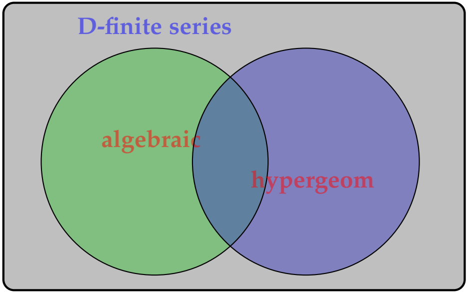
Our goal: Use computer algebra to give computational answers.

Important Classes of Univariate Power Series





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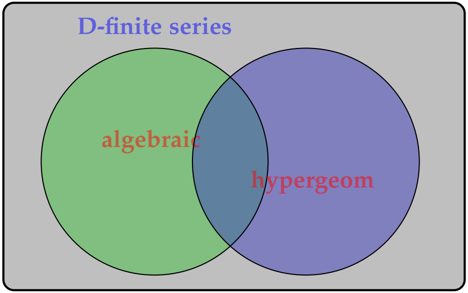

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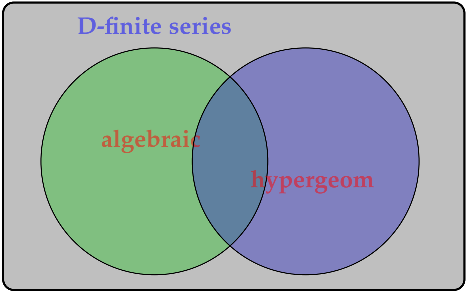

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Hypergeometric: $S(t) = \sum_{n=0}^{\infty} s_n t^n$ such that $\frac{s_{n+1}}{s_n} \in \mathbb{Q}(n)$. E.g.,

$$ {}_2F_1\left(\begin{matrix} a\ b \\ c \end{matrix} \middle| t\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \qquad (a)_n = a(a+1)\cdots(a+n-1). $$

Small-Step Models of Interest

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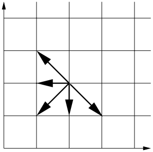

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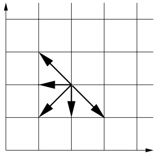

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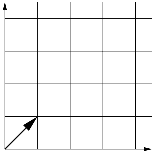

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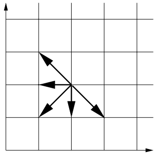

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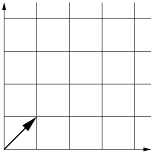

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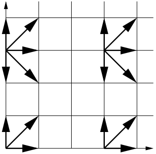

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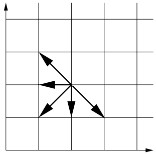

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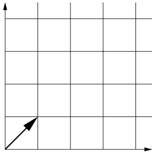

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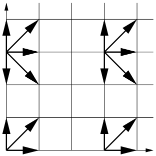

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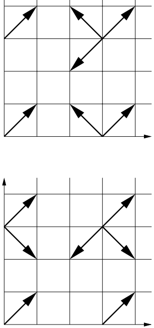

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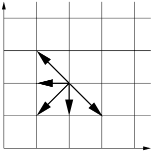

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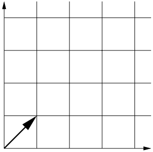

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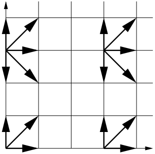

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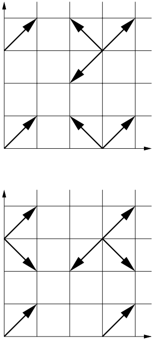

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One is left with 79 interesting distinct models.

Table of All Conjectured D-Finite $F(t;1,1)$ [Bostan & Kauers 2009]

	OEIS	$\mathfrak{S}$	alg	equiv		OEIS	$\mathfrak{S}$	alg	equiv
1	A005566		N	$\frac{4}{\pi}\frac{4^n}{n}$	13	A151275		N	$\frac{12\sqrt{30}}{\pi}\frac{(2\sqrt{6})^n}{n^2}$
2	A018224		N	$\frac{2}{\pi}\frac{4^n}{n}$	14	A151314		N	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi}\frac{(2C)^n}{n^2}$
3	A151312		N	$\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$	15	A151255		N	$\frac{24\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2}$
4	A151331		N	$\frac{8}{3\pi}\frac{8^n}{n}$	16	A151287		N	$\frac{2\sqrt{2}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$
5	A151266		N	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{1/2}}$	17	A001006		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{3/2}}$
6	A151307		N	$\frac{1}{2}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	18	A129400		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{6^n}{n^{3/2}}$
7	A151291		N	$\frac{4}{3\sqrt{\pi}}\frac{4^n}{n^{1/2}}$	19	A005558		N	$\frac{8}{\pi}\frac{4^n}{n^2}$
8	A151326		N	$\frac{2}{\sqrt{3\pi}}\frac{6^n}{n^{1/2}}$					
9	A151302		N	$\frac{1}{3}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	20	A151265		Y	$\frac{2\sqrt{2}}{\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
10	A151329		N	$\frac{1}{3}\sqrt{\frac{7}{3\pi}}\frac{7^n}{n^{1/2}}$	21	A151278		Y	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
11	A151261		N	$\frac{12\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^2}$	22	A151323		Y	$\frac{\sqrt{2}3^{3/4}}{\Gamma(1/4)}\frac{6^n}{n^{3/4}}$
12	A151297		N	$\frac{\sqrt{3}B^{7/2}}{2\pi}\frac{(2B)^n}{n^2}$	23	A060900		Y	$\frac{4\sqrt{3}}{3\Gamma(1/3)}\frac{4^n}{n^{2/3}}$

$A = 1+\sqrt{2},\ B = 1+\sqrt{3},\ C = 1+\sqrt{6},\ \lambda = 7+3\sqrt{6},\ \mu = \sqrt{\frac{4\sqrt{6}-1}{19}}$

▷ Computerized discovery by enumeration + Hermite–Padé + LLL/PSLQ.

PROVE THIS TABLE!

Previous Work on the 23 Cases

▷ Human proof of D-finiteness/algebraicity for cases 1–22 in [Bousquet-Mélou & Mishna, 2010]:

- based on averaging over a group of rational invariant transformations,
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▷ Human proofs of asymptotics $f_n \sim \kappa n^\alpha \rho^n$:

- ρ for all cases in [Fayolle & Raschel, 2012];
- (α, ρ) for cases 1–4,17–23 (zero drift) using [Denisov & Wachtel, 2013];
- (κ, α, ρ) for cases 1–4 (2 axes of sym.) in [Melczer & Mishna, 2014];
- (κ, α, ρ) for cases 17–22 in [Bousquet-Mélou & Mishna, 2010];
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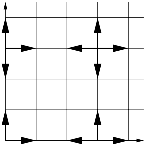
D-Finiteness via the Finite Group: an Example



$J = -\frac{1}{t} + \sum_{(i,j)\in\mathfrak{S}} x^i y^j = x + \frac{1}{x} + y + \frac{1}{y} - \frac{1}{t}$ is invariant under the change of (x, y) into, respectively:

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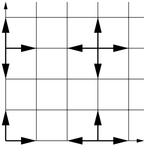

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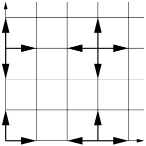

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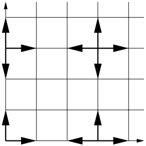

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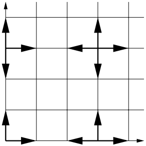

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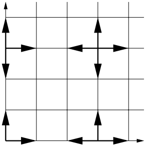

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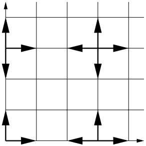
\end{aligned}$$

Summing up yields:

$$\sum_{\theta\in\mathcal{G}} (-1)^\theta \theta\big(xyt\,F(t;x,y)\big) = \frac{-xy + \frac{1}{x}y - \frac{1}{x}\frac{1}{y} + x\frac{1}{y}}{J(t;x,y)}$$

D-Finiteness via the Finite Group: an Example



$J = -\frac{1}{t} + \sum_{(i,j)\in\mathfrak{S}} x^i y^j = x + \frac{1}{x} + y + \frac{1}{y} - \frac{1}{t}$ is invariant under the change of (x, y) into, respectively:

$$\left(\frac{1}{x}, y\right), \left(\frac{1}{x}, \frac{1}{y}\right), \left(x, \frac{1}{y}\right).$$

"Kernel equation":

$$\begin{aligned}
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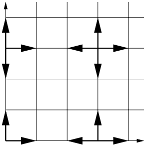

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Cases 1–19 are D-Finite

Theorem [Bousquet-Mélou & Mishna, 2010]

Let $\mathfrak{S}$ be one of the step sets 1–19. Then, the invariant group $\mathcal{G}$ is finite and:

$$xyt\,F(t;x,y) = [x^>][y^>]\,\frac{\sum_{\theta\in\mathcal{G}}(-1)^{\theta}\theta(xy)}{J(t;x,y)}.$$

In particular, $F(t;x,y)$ is D-finite.

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Proof: Use [Lipshitz'88] (*"The diagonal of a D-finite power series is D-finite"*) for positive parts of D-finite series.

▷ Constructive proof, but it leads to a highly inefficient algorithm to get an ODE for $F(t;x,y)$.

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TOO LARGE TO BE MERELY WRITTEN!

Explicit Expressions for the Cases 1–19

Theorem [Bostan-Chyzak-van Hoeij-Kauers-Pech, 2014]

Let $\mathfrak{S}$ be one of the step sets 1–19. Then, the generating series $F(t;x,y)$ is expressible using iterated integrals of ${}_2F_1$ expressions.

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Theorem [Bostan-Chyzak-van Hoeij-Kauers-Pech, 2014]

Let $\mathfrak{S}$ be one of the step sets 1–19. Then, the generating series $F(t;1,1)$ is expressible using iterated integrals of ${}_2F_1$ expressions.

Example: King walks in the quarter plane (A025595)

$$F(t;1,1) = \frac{1}{t}\int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2} \\ & 2\end{matrix} \,\middle|\, \frac{16x(1+x)}{(1+4x)^2}\right) dx$$
$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

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▷ Proof uses Creative telescoping, ODE factorization, ODE solving:

1. (New) If $R = \sum_\theta \frac{(-1)^\theta\,\theta(xy)}{J(t;x,y)}$, then $F = \operatorname{Res}_u\left(\operatorname{Res}_v H\right)$, for $H = \frac{R(t;1/u,1/v)}{(1-xu)(1-yv)}$.

2. (New) If $P \in \mathbb{Q}(x,y)[t]\langle\partial_t\rangle$ and $U, V \in \mathbb{Q}(x,y,u,v,t)$ such that $L(H) = \partial_u U + \partial_v V$, then $L(F(t;x,y)) = 0$.
   Use creative telescoping for finding L.

3. Factor L as $L_2 \cdot P_1 \cdots P_t$, where L_2 has order 2 and the P_i have order 1.
   Solve L_2 in terms of ${}_2F_1$s and deduce F.

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   Use creative telescoping for finding L.
   Works in practice with early evaluation $(x,y) = (1,1)$, but not for symbolic (x,y).
   Works also for $(0,0)$, $(x,0)$, and $(0,y)$!
3. Factor L as $L_2 \cdot P_1 \cdots P_t$, where L_2 has order 2 and the P_i have order 1.
   Solve L_2 in terms of ${}_2F_1$s and deduce F.

For $F(t;x,y)$, run whole process for $F(t;0,0)$, $F(t;x,0)$, and $F(t;0,y)$, then substitute into Kernel equation!

Hypergeometric Series Occurring in Explicit Expressions for $F(t;1,1)$

	hyp$_1$	hyp$_2$	w		hyp$_1$	hyp$_2$	w
1	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ 1\end{matrix}\middle\| w\right)$	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{3}{2}\\ 2\end{matrix}\middle\| w\right)$	$16t^2$	10	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\middle\| w\right)$	${}_2F_1\left(\begin{matrix}\frac{9}{4} & \frac{11}{4}\\ 3\end{matrix}\middle\| w\right)$	$\frac{64(t^2+t+1)t^2}{(12t^2+1)^2}$
2	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ 1\end{matrix}\middle\| w\right)$		$16t^2$	11	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{3}{2}\\ 2\end{matrix}\middle\| w\right)$	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{5}{2}\\ 3\end{matrix}\middle\| w\right)$	$\frac{16t^2}{4t^2+1}$
3	${}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2}\\ 2\end{matrix}\middle\| w\right)$		$\frac{16t}{(2t+1)(6t+1)}$	12	${}_2F_1\left(\begin{matrix}\frac{5}{4} & \frac{7}{4}\\ 1\end{matrix}\middle\| w\right)$	${}_2F_1\left(\begin{matrix}\frac{5}{4} & \frac{7}{4}\\ 2\end{matrix}\middle\| w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$
4	${}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2}\\ 2\end{matrix}\middle\| w\right)$		$\frac{16t(1+t)}{(1+4t)^2}$	13	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\middle\| w\right)$	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 3\end{matrix}\middle\| w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$
5	${}_2F_1\left(\begin{matrix}\frac{3}{4} & \frac{5}{4}\\ 1\end{matrix}\middle\| w\right)$	${}_2F_1\left(\begin{matrix}\frac{5}{4} & \frac{7}{4}\\ 2\end{matrix}\middle\| w\right)$	$64t^4$	14	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\middle\| w\right)$	${}_2F_1\left(\begin{matrix}\frac{9}{4} & \frac{11}{4}\\ 3\end{matrix}\middle\| w\right)$	$\frac{64(t^2+t+1)t^2}{(12t^2+1)^2}$
6	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\middle\| w\right)$	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 3\end{matrix}\middle\| w\right)$	$\frac{64t^3(1+t)}{(1-4t^2)^2}$	15	${}_2F_1\left(\begin{matrix}\frac{1}{4} & \frac{3}{4}\\ 1\end{matrix}\middle\| w\right)$	${}_2F_1\left(\begin{matrix}\frac{3}{4} & \frac{5}{4}\\ 2\end{matrix}\middle\| w\right)$	$64t^4$
7	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ 1\end{matrix}\middle\| w\right)$	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{3}{2}\\ 1\end{matrix}\middle\| w\right)$	$\frac{16t^2}{4t^2+1}$	16	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\middle\| w\right)$	${}_2F_1\left(\begin{matrix}\frac{9}{4} & \frac{11}{4}\\ 3\end{matrix}\middle\| w\right)$	$\frac{64t^3(1+t)}{(1-4t^2)^2}$
8	${}_2F_1\left(\begin{matrix}\frac{5}{4} & \frac{7}{4}\\ 2\end{matrix}\middle\| w\right)$	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\middle\| w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$				
9	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\middle\| w\right)$	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 3\end{matrix}\middle\| w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$	19	${}_2F_1\left(\begin{matrix}-\frac{1}{2} & \frac{1}{2}\\ 1\end{matrix}\middle\| w\right)$	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ 2\end{matrix}\middle\| w\right)$	$16t^2$

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	hyp$_1$	hyp$_2$	w		hyp$_1$	hyp$_2$	w
1	${}_2F_1\left(\begin{matrix}\frac12 & \frac12\\ 1\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac12 & \frac32\\ 2\end{matrix}\middle\vert w\right)$	$16t^2$	10	${}_2F_1\left(\begin{matrix}\frac74 & \frac94\\ 2\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac94 & \frac{11}4\\ 3\end{matrix}\middle\vert w\right)$	$\frac{64(t^2+t+1)t^2}{(12t^2+1)^2}$
2	${}_2F_1\left(\begin{matrix}\frac12 & \frac12\\ 1\end{matrix}\middle\vert w\right)$		$16t^2$	11	${}_2F_1\left(\begin{matrix}\frac12 & \frac32\\ 2\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac12 & \frac52\\ 3\end{matrix}\middle\vert w\right)$	$\frac{16t^2}{4t^2+1}$
3	${}_2F_1\left(\begin{matrix}\frac32 & \frac32\\ 2\end{matrix}\middle\vert w\right)$		$\frac{16t}{(2t+1)(6t+1)}$	12	${}_2F_1\left(\begin{matrix}\frac54 & \frac74\\ 1\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac54 & \frac74\\ 2\end{matrix}\middle\vert w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$
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5	${}_2F_1\left(\begin{matrix}\frac34 & \frac54\\ 1\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac54 & \frac74\\ 2\end{matrix}\middle\vert w\right)$	$64t^4$	14	${}_2F_1\left(\begin{matrix}\frac74 & \frac94\\ 2\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac94 & \frac{11}4\\ 3\end{matrix}\middle\vert w\right)$	$\frac{64(t^2+t+1)t^2}{(12t^2+1)^2}$
6	${}_2F_1\left(\begin{matrix}\frac74 & \frac94\\ 2\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac74 & \frac94\\ 3\end{matrix}\middle\vert w\right)$	$\frac{64t^3(1+t)}{(1-4t^2)^2}$	15	${}_2F_1\left(\begin{matrix}\frac14 & \frac34\\ 1\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac34 & \frac54\\ 2\end{matrix}\middle\vert w\right)$	$64t^4$
7	${}_2F_1\left(\begin{matrix}\frac12 & \frac12\\ 1\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac12 & \frac32\\ 1\end{matrix}\middle\vert w\right)$	$\frac{16t^2}{4t^2+1}$	16	${}_2F_1\left(\begin{matrix}\frac74 & \frac94\\ 2\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac94 & \frac{11}4\\ 3\end{matrix}\middle\vert w\right)$	$\frac{64t^3(1+t)}{(1-4t^2)^2}$
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Observation: Pairs of related hyps $+\,{}_2F_1\left(\begin{matrix}a\ b\\ c\end{matrix}\middle\vert w\right)$ with $m = c-(a+b) \in \mathbb{Z}$.

Proofs of Algebraicity/Transcendence of $F(t; x, y)$ and $F(t; 1, 1)$

Theorem

- In cases 1–19, $F(t; x, y)$ is transcendental since $F(t; 0, 0)$ is.
- In cases 1–16 and 19, $F(t; 1, 1)$ is transcendental.
- Specific simplifications prove algebraicity of $F(t; 1, 1)$ in cases 17–18.

Proof: Define $G = (P_1 \cdots P_t)(F)$ so that $L_2(G) = 0$.

- F is algebraic $\Longrightarrow$ G is algebraic.
- Computing a few coefficients of G shows that this is not 0 on all cases of interest.
- Applying Kovacic's algorithm to L_2 decides whether L_2 has nonzero algebraic solutions.

Coefficient Asymptotics $\kappa n^\alpha \rho^n$ *(in progress)*

Local theory of D-finite functions $\longrightarrow$
Systematic method for coefficient asymptotics
(Flajolet and Odlyzko's singularity analysis)

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \qquad \longrightarrow \qquad f_n \sim \dots$$

- Determine dominant singularities of the *complex-analytic function* f.
- Find asymptotic expansion

$$f(z) =_{z \to s} \sum_{\alpha,\gamma} c_{\alpha,\gamma} (s-z)^\alpha \left(\ln \frac{1}{s-z} \right)^\gamma \tag{1}$$

- Syntactic transfer into an asymptotic expansion for f_n

Three Formulas from (DLMF 15.8) on ${}_2F_1$s

▷ To ensure that $c - a - b \in \mathbb{N}$: for $m \in \mathbb{N}$,

$$ {}_2F_1\left(\begin{matrix} a \quad b \\ a+b-m \end{matrix} \middle| z\right) = (1-z)^{-m} {}_2F_1\left(\begin{matrix} a-m \quad b-m \\ (a-m)+(b-m)+m \end{matrix} \middle| z\right) $$

▷ To ensure bring $-\infty$ at 1: for $z < 1/2$,

$$ {}_2F_1\left(\begin{matrix} a \quad b \\ \frac{1}{2}(a+b+1) \end{matrix} \middle| z\right) = (1-2z)^{-a} {}_2F_1\left(\begin{matrix} \frac{1}{2}a \quad \frac{1}{2}a+\frac{1}{2} \\ \frac{1}{2}(a+b+1) \end{matrix} \middle| \frac{4z(z-1)}{(1-2z)^2}\right) $$

▷ Local logarithmic behaviour at 1: for $m \in \mathbb{N},\ z \in D(1,1) \setminus [0,1]$,

$$ {}_2F_1\left(\begin{matrix} a \quad b \\ a+b+m \end{matrix} \middle| z\right) = $$

$$ \text{polynomial of degree } m-1 \text{ in } 1-z $$
$$ + \text{term in } (1-z)^m \ln(1-z) + \text{higher order terms} $$

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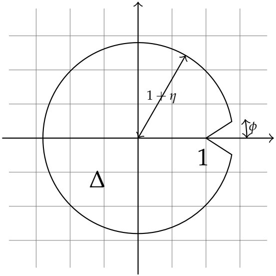
$$ {}_2F_1\left(\begin{matrix} a \quad b \\ \frac{1}{2}(a+b+1) \end{matrix}\,\middle|\, z\right) = (1-2z)^{-a} {}_2F_1\left(\begin{matrix} \frac{1}{2}a \quad \frac{1}{2}a+\frac{1}{2} \\ \frac{1}{2}(a+b+1) \end{matrix}\,\middle|\, \frac{4z(z-1)}{(1-2z)^2}\right) $$

▷ Local logarithmic behaviour at 1: for $m \in \mathbb{N}$, $z \in D(1,1) \setminus [0,1]$,

$$ \begin{aligned} {}_2F_1\left(\begin{matrix} a \quad b \\ a+b+m \end{matrix}\,\middle|\, z\right) &= \\ &\frac{1}{\Gamma(a+m)\Gamma(b+m)} \sum_{k=0}^{m-1} (-1)^k \frac{(a)_k (b)_k (m-k-1)!}{k!} (1-z)^k \\ &- (-1)^m \frac{(1-z)^m}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{(a+m)_k (b+m)_k}{k!(k+m)!} (1-z)^k \Big(\ln(1-z) \\ &\quad - \psi(k+1) - \psi(k+m+1) + \psi(a+k+m) + \psi(b+k+m)\Big) \end{aligned} $$

Transfer Theorems [Flajolet & Odlyzko 1990]



For $f(z) = \sum_{n=0}^{\infty} f_n z^n$ analytic in $\Delta \setminus \{1\}$:

$f(z)$	f_n	assumptions
$O\big((1-z)^{\alpha}\big)$	$O\big(n^{-(\alpha+1)}\big)$	$\alpha \in \mathbb{R}$
$o\big((1-z)^{\alpha}\big)$	$o\big(n^{-(\alpha+1)}\big)$	$\alpha \in \mathbb{R}$
$\sim C(1-z)^{\alpha}$	$\sim \frac{Cn^{-(\alpha+1)}}{\Gamma(-\alpha)}$	$\alpha \in \mathbb{R} \setminus \mathbb{N}$
$\sum_{j=0}^{m-1} c_j(1-z)^{\alpha_j} + O\big((1-z)^A\big)$	$\sum_{j=0}^{m-1} \frac{c_j n^{-(\alpha_j+1)}}{\Gamma(-\alpha_j)} + O\big(n^{-(A+1)}\big)$	$\alpha_1 \leq \cdots \leq \alpha_{m-1} < A$
$O\big((1-z)^{\alpha}(\ln(1-z)^{-1})^{\gamma}\big)$	$O\big(n^{-(\alpha+1)}(\ln n)^{\gamma}\big)$	$\alpha, \gamma \in \mathbb{R}$
$o\big((1-z)^{\alpha}(\ln(1-z)^{-1})^{\gamma}\big)$	$o\big(n^{-(\alpha+1)}(\ln n)^{\gamma}\big)$	$\alpha, \gamma \in \mathbb{R}$
$\sim C(1-z)^{\alpha}(\ln(1-z)^{-1})^{\gamma}$	$\sim \frac{Cn^{-(\alpha+1)}(\ln n)^{\gamma}}{\Gamma(-\alpha)}$	$\alpha, \gamma \in \mathbb{R} \setminus \mathbb{N}$
$\vdots$	$\vdots$	

$$Q = \frac{1}{t}\int f \quad \text{for} \quad f = (1-2t)(1+2t)^{-3/2}(1+6t)^{-3/2}{}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2} \\ & 2\end{matrix}\,\middle|\, w\right)$$

$$\text{where} \quad w = \frac{16t}{(1+2t)(1+6t)}$$

Singularities: $\frac{1}{2}$, $-\frac{1}{2}$, $-\frac{1}{6}$, $w = 1$, $w = \infty \longrightarrow$ Dominant singularities = $\pm\frac{1}{6}$.

$$f(t) \sim_{t \to \frac{1}{6}^-} \frac{\sqrt{6}}{\pi}(1-6t)^{-1} \qquad \longrightarrow \qquad \frac{\sqrt{6}}{\pi}6^n$$

$$f(t) \sim_{t \to -\frac{1}{6}^+} \frac{\sqrt{6}}{4\pi}\ln(1+6t) \qquad \longrightarrow \qquad \frac{\sqrt{6}}{4\pi}\frac{6^n}{n}$$

One Example:  at $(1,1)$

$$Q = \frac{1}{t}\int f \quad \text{for} \quad f = (1-2t)(1+2t)^{-3/2}(1+6t)^{-3/2}\,{}_2F_1\left(\left.\begin{matrix}\frac{3}{2} & \frac{3}{2} \\ & 2\end{matrix}\,\right|\, w\right)$$

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$$f \quad \longrightarrow \quad f_n \sim \frac{\sqrt{6}}{\pi}6^n$$

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$$\int f \quad \longrightarrow \quad f_n \sim \frac{\sqrt{6}}{\pi} \frac{6^{n-1}}{n}$$

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$$\frac{1}{t}\int f \quad \longrightarrow \quad f_n \sim \frac{\sqrt{6}}{\pi}\frac{6^n}{n+1}$$

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$$\frac{1}{t}\int f \quad \longrightarrow \quad f_n \sim \frac{\sqrt{6}}{\pi}\frac{6^n}{n}$$

Conclusions

Summary: three kinds of conjectures now proved:

- differential operators that witness D-finiteness,
- algebraic vs transcendental nature of series,
- asymptotics of coefficients (in progress).