Computing necessary integrability conditions for planar parametrized homogeneous potentials

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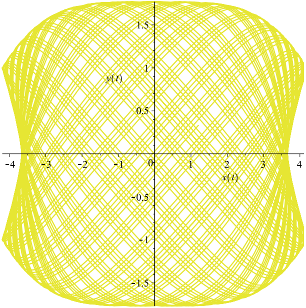
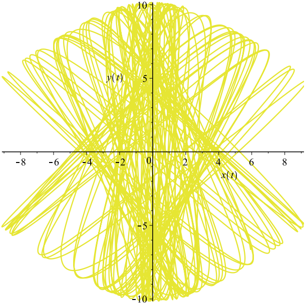
Motion of a point mass under a force field: potential V, equations

$$\ddot{q}_1 = -\frac{\partial}{\partial q_1} V(q) \qquad \ddot{q}_2 = -\frac{\partial}{\partial q_2} V(q)$$

Liouville Arnold Theorem (1980):
Integrability $\Leftrightarrow$ quasi-periodic motion $\Leftrightarrow$ exactly solvable system





Integrability analysis: Darboux search for integrable systems (1901), Hietarinta list of integrable systems (1987).

Problem

$V \in \mathbb{Q}(\mathbf{a})(q_1, q_2)$ homogeneous in q_1, q_2 with parameters $\mathbf{a}$. Find all parameters values $\mathbf{a}$ such that this potential is integrable.

Integrability $\Leftrightarrow$ Regular motion $\Rightarrow$ exceptional phenomenon!

Our goal

Algorithm

Input: $V \in \mathbb{Q}(\mathbf{a})(q_1, q_2)$ homogeneous in q_1, q_2 with parameters $\mathbf{a}$.
Output: Non-trivial necessary conditions on $\mathbf{a}$ for integrability of V.

What is already done?

- Ziglin (1983): Theoretical conditions for integrability
- Morales-Ramis (1997): Explicit conditions for integrability
- Maciejewski-Przybylska (2000): Universal relations allowing to test these conditions in the parametrized case

What remains to be done?

- handle automatically singular parameter's values
- handle multiplicity of roots/poles and degenerated asymptotics
- precise when and what happens when the universal relation does not hold

Our contributions

- Rewriting Morales-Ramis integrability conditions and Maciejewski-Przybylska relation under a simpler form
- Ability to deal with singular specializations of the parameters
- Complete automation of the computation of integrability conditions through Groebner bases
- Explicit application and discovery of new candidates for integrability

- **Integrability conditions** (Morales-Ramis): the eigenvalues of $\nabla^2 V(c)$ at Darboux points, i.e. solutions of

$$\frac{\partial V}{\partial q_1}(c_1, c_2) = kc_1 \qquad \frac{\partial V}{\partial q_2}(c_1, c_2) = kc_2$$

  should belong to a (completely explicit) discrete set E_k

- **Universal relation**: there exists a relation R between eigenvalues independent of parameters (Maciejewski-Przybylska)

General strategy

- Compute the relation R.
- Solve R in E_k, get finitely many solutions.
- For each solution, rewrite the conditions on eigenvalues as polynomial conditions on **a**.

Step 1: Polar coordinates

$$V = (\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3)q_1^3 + i(3\mathbf{a}_1 - \mathbf{a}_3 + \mathbf{a}_2)q_1^2 q_2 +$$

$$(-3\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3)q_1 q_2^2 - i(\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3)q_2^3$$

Write V in polar coordinates

$$V = r^3 F(e^{i\theta}) \qquad F(z) = \mathbf{a}_1 z^3 + \mathbf{a}_2 z + \mathbf{a}_3/z$$

For $c = (r\cos\theta, r\sin\theta)$ a Darboux point, we have

$$F'(e^{i\theta}) = 0, \quad F(e^{i\theta}) \neq 0$$

$$Sp(\nabla^2 V(c)) = \left\{6, 3 - \frac{(e^{i\theta})^2 F''(e^{i\theta})}{F(e^{i\theta})}\right\} = \{6, \lambda\}$$

Step 2: Computation of the relation

The relation R reads

$$\sum_{\substack{\lambda \\ \text{eigenvalue}}} \frac{1}{\lambda-3} = \frac{1}{k_0} - \frac{1}{k_\infty}$$

where

$$F(z) \underset{z\to 0}{\sim} a_0 z^{k_0} \quad F(z) \underset{z\to\infty}{\sim} a_0 z^{k_\infty}$$

3 important parameters:
number of eigenvalues p, asymptotic constants k_0, k_∞

Parity of F allows simplifications:
p is even, eigenvalues λ come by pairs

Depending on specialization of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, 8 possibilities

$$\begin{tikzcd}
 & \boxed{p=4,\, k_\infty=3,\, k_0=-1} \arrow[dl] \arrow[d] \arrow[dr] & \\
\boxed{p=2,\, k_\infty=1,\, k_0=-1} \arrow[d] \arrow[dr] & \boxed{p=2,\, k_\infty=3,\, k_0=-1} \arrow[dl] \arrow[dr] & \boxed{p=2,\, k_\infty=3,\, k_0=1} \arrow[dl] \arrow[d] \\
\boxed{p=0,\, k_\infty=1,\, k_0=1} \arrow[dr] & \boxed{p=0,\, k_\infty=1,\, k_0=1} \arrow[d] & \boxed{p=0,\, k_\infty=3,\, k_0=3} \arrow[dl] \\
 & \boxed{F=0} &
\end{tikzcd}$$

Step 3: Solving the relation

If $p = 0$, no integrability conditions.

If $p = 2$: $\frac{2}{\lambda_1 - 3} = -2$ or $-\frac{4}{3}$ or $\frac{2}{3}$
One solution in E_3: $\lambda_1 = 6$

If $p = 4$: $\frac{2}{\lambda_1 - 3} + \frac{2}{\lambda_2 - 3} = -\frac{4}{3}$
To solve this equation in E_3: bounding $\min(\lambda_1, \lambda_2)$

$$\min(\lambda_1, \lambda_2) \leq 3 \Rightarrow \min(\lambda_1, \lambda_2) \in \{0, 1\}$$

Recursively solve the equation with one unknown less
$\Rightarrow (\lambda_1, \lambda_2) = (0, 0)$

Step 4: Conditions on a

Rewriting these conditions as polynomial conditions on a

If $p=2, k_\infty=3, k_0=1$, then

- $\mathbf{a}_3=0, \mathbf{a}_1\neq 0, \mathbf{a}_2\neq 0$ (multiplicity & asymptotics)
- $3-\frac{z^2F''(z)}{F(z)}=6$ for $F'(z)=0, F(z)\neq 0, z\neq 0$ (integrability)

Second condition rewrites as polynomial divisibility

$$\mathrm{numer}\left(\frac{F'}{F}\right) \mid \mathrm{numer}\left(3-\frac{z^2F''(z)}{F(z)}-6\right) \quad \Rightarrow \mathbf{a}_3=0$$

If $p = 4, k_\infty = 3, k_0 = -1$, then

- $\mathbf{a}_3 \neq 0, \mathbf{a}_1 \neq 0, \mathbf{a}_2 \neq 0, \mathbf{a}_2^2 - 4\mathbf{a}_1\mathbf{a}_3 \neq 0$ (multiplicity & asymptotics)
- $3 - \frac{z^2 F''(z)}{F(z)} = 0$ for $F'(z) = 0, F(z) \neq 0, z \neq 0$ (integrability)

Second condition rewrites as polynomial divisibility

$$\text{numer}\left(\frac{F'}{F}\right) \mid \text{numer}\left(3 - \frac{z^2 F''(z)}{F(z)}\right) \quad \Rightarrow \mathbf{a}_2 = 0$$

$\Rightarrow$ The output is $[\mathbf{a}_1, \mathbf{a}_2], [\mathbf{a}_1, \mathbf{a}_3], [\mathbf{a}_2, \mathbf{a}_3], [\mathbf{a}_2], [\mathbf{a}_3]$

Implementation in Maple 17.

Main tool: Elimination in polynomial ideals through Fgb

Testing the condition on eigenvalues $\Leftrightarrow$ Euclidean division of univariate polynomials

The collinear three body problem:
Three bodies on the line interacting by gravity.

After reduction, potential of the form

$$V = \frac{1}{\mathbf{a}_1 q_1 + \mathbf{a}_2 q_2} + \frac{1}{\mathbf{a}_3 q_1 + a_4 q_2} + \frac{1}{a_5 q_1 + a_6 q_2}$$

Thanks to the algorithm

Theorem

The collinear three body problem with non zero masses is non-integrable.

Conclusion

Other integrability conditions rely on eigenvalues and higher order derivatives of V
$\Rightarrow$ these can be easily included in the algorithm

Most of the computation time on the elimination $\Rightarrow$ search a better representation to improve times

Implementation of additional conditions $\Rightarrow$ probably only zero dimensional ideal would appear $\Rightarrow$ improved times