# Diagonals and Walks

Alin Bostan[1] Louis Dumont[1] Bruno Salvy[2]

[1]INRIA, SpecFun

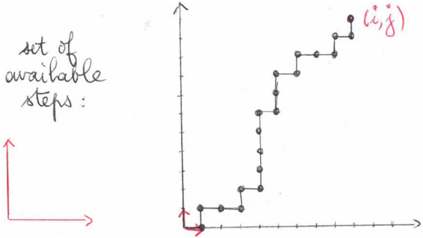
[2]INRIA, AriC

November 4, 2014

# Introduction

# Motivations

In combinatorics:
Lattice walks



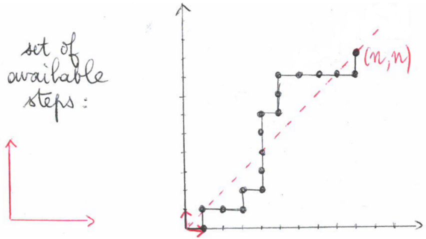

$w_{i,j}$: number of walks that end at $(i,j)$ $\longrightarrow$ generating power series: $W(X,Y) = \sum_{i,j\geq 0} w_{i,j} X^i Y^j$

$w_{n,n}$: number of walks that end on the diagonal at $(n,n)$ $\longrightarrow$ diagonal series: $\operatorname{diag} W(T) = \sum_{n\geq 0} w_{n,n} T^n$

Diagonals also appear in statistical physics, number theory,...

# Motivations

In combinatorics:
Lattice walks




$w_{i,j}$: number of walks that end at $(i,j)$ $\longrightarrow$ generating power series: $W(X,Y) = \sum_{i,j\geq 0} w_{i,j} X^i Y^j$

$w_{n,n}$: number of walks that end on the diagonal at $(n,n)$ $\longrightarrow$ diagonal series: $\operatorname{diag} W(T) = \sum_{n\geq 0} w_{n,n} T^n$

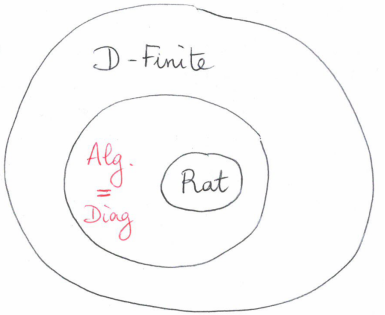
Diagonals also appear in statistical physics, number theory,...

# Diagonals

$$F(X,Y) = \frac{1}{1-X-Y} = \sum_{i,j\geq 0} \binom{i+j}{i} X^i Y^j$$

$$\mathrm{diag}(F) = \sum_{n\geq 0} \binom{2n}{n} T^n = \frac{1}{\sqrt{1-4T}}$$

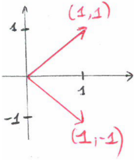
| | | | | |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 1 | 2 | 3 | 4 | 5 |
| 1 | 3 | 6 | 10 | 15 |
| 1 | 4 | 10 | 20 | 35 |
| 1 | 5 | 15 | 35 | 70 |

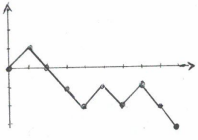



Abel ($\approx$ 1830): Alg $\subset$ D-Finite
Furstenberg (1967): Diag = Alg

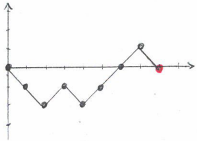
# 1-dimensional walks

Set of steps of the form $(1, a)$, $a \in \mathbb{Z}$
Example : Dyck Paths

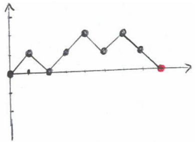


Walks

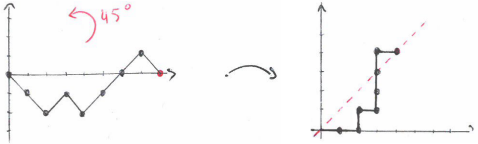


Bridges



Excursions

$B(T) = \sum_{n \geq 0} b_n T^n$, $b_n$ : number of bridges of length $n$
$B(T)$ can be expressed as a diagonal.

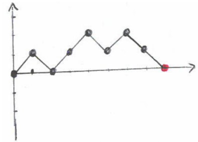


# Questions

## Question 1

How hard is it to compute a polynomial equation satisfied by the diagonal of a bivariate rational function ?

$E(T) = \sum_{n\geq 0} e_n T^n$, $e_n$ : number of excursions of length $n$



## Question 2

How to compute $E(T) \mod T^N$ for a given $N$ ?

## Answers

### Theorem 1 (B., D., S.)

*Generically:*

- *the minimal polynomial equation can be computed in time that is quasi-linear with respect to its size.*
- *the minimal polynomial equation is exponentially bigger than the initial rational function ($16^d$ vs $d^2$, where $d$ is the degree)*

| $d$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\deg_T P, \deg_Y P$ | (2, 2) | (16, 6) | (108, 20) | (640, 70) |

### Theorem 2 (B., D., S.)

*$E(T) \mod T^N$ can be computed in $\tilde{O}(N)$ arithmetic operations, with a fairly inexpensive pre-computation.*

# Diagonals

How to compute a polynomial equation satisfied by the diagonal of a bivariate rational function ?

# Algebraic equation for the diagonal

Ingredients of the algorithm:

- Partial fraction decomposition: the diagonal is a sum of residues of a rational function
- Rothstein-Trager resultant (1976): algebraic equation that cancels all residues $\alpha_1(T), \alpha_2(T), \ldots, \alpha_d(T)$
- Main difficulty: if $P = \prod_{i=1}^{d} (Y - \alpha_i)$ is known, efficiently compute the polynomial

$$P_k = \prod_{\{i_1, i_2, \ldots, i_k\}} (Y - (\alpha_{i_1} + \alpha_{i_2} + \ldots + \alpha_{i_k}))$$

for a given $k \leq d$

# Newton sums

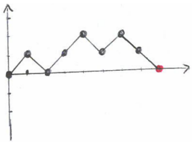
Most important part of our algorithm: a way around the main difficulty using **Newton sums**.

$$P = \prod_{i=1}^{d} (Y - \alpha_i) \ \leftrightarrow \ \mathcal{N}(P) = \sum_{n \geq 0} (\alpha_1^n + \alpha_2^n + \ldots + \alpha_d^n) \frac{Y^n}{n!}$$

- $P \leftrightarrow \mathcal{N}(P)$ is well-known and effective
- $\mathcal{N}(P_k)$ can be computed from $\mathcal{N}(P)$
- $P_k$ is then recovered from $\mathcal{N}(P_k)$

# Walks

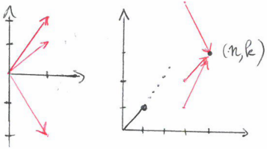
How to compute $E(T) \mod T^N$ for a given $N$ ?



# Excursions: naive method

- $w_{n,k}$ : number of walks that stay in the upper half plane, of length $n$ and ending at height $k$
- $e_n = w_{n,0}$ ($e_n$: number of excursions of length $n$)
- $w_{n,k}$ satisfies a linear recurrence relation with constant coefficients :

$$w_{n,k} = \sum_{(1,a)\in\mathcal{S}} w_{n-1,k-a},$$




where $\mathcal{S}$ is the set of available steps

- Algorithm with $O(N^2)$ arithmetic operations and no pre-computation

# Excursions: more efficient algorithm

**Idea**

(Banderier, Flajolet, 2002) Use the fact that $E(T)$ is algebraic to find a better recurrence relation.

- recurrence for $e_n$ (1 index) instead of $w_{n,k}$ (2 indices) $\rightarrow O(N)$ operations
- **But** linear complexity at the cost of pre-computations:
  - the algebraic equation, which is exponentially big in d (Bousquet-Mélou, 2008)
  - algebraic eqn $\rightarrow$ differential eqn (also exponentially big) (Bostan, Chyzak, Lecerf, Salvy, Schost, 2007)
  - initial conditions of the recurrence (exponentially many)

Algorithm with $O(N)$ operations and pre-computation of an exponential size in d equation

# Excursions: new algorithm

## Ideas

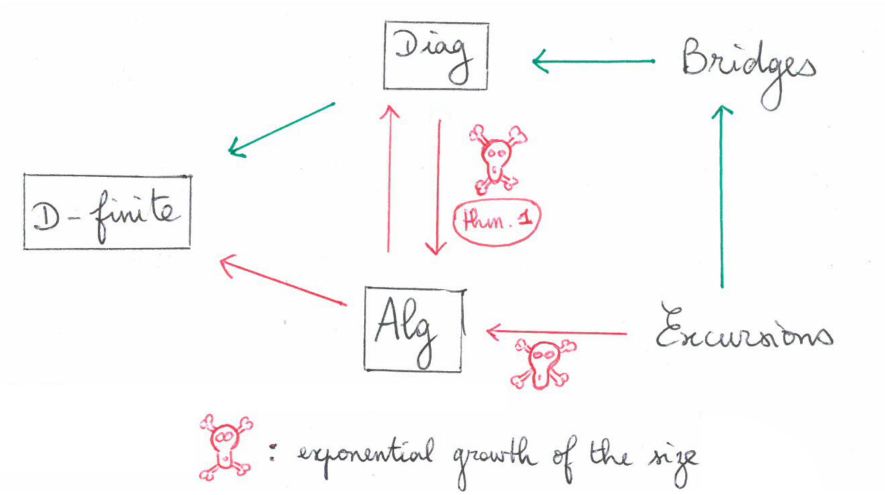
- $E(T) = \exp \int_0^T \frac{B(t)-1}{t} \mathrm{d}t$ (Banderier, Flajolet 2002). Recover $E(T)$ from $B(T)$ using this formula.
- Diagonals (including $B(T)$) satisfy small (polynomial size) differential equations (Bostan, Chen, Chyzak, Li, 2010)).

Method:

- Fast computation of a differential equation for $B$ using Hermite reduction (loc. cit.)
- Calculate $B \mod T^N$ with this equation $O(N)$
- Apply the formula using Newton iteration $\tilde{O}(N)$

Algorithm with $\tilde{O}(N)$ operations and pre-computation of a polynomial size in d equation

# Conclusion


Diag
Bridges
D-finite
Thm. 1
Alg
Excursions
: exponential growth of the size