Efficient algorithms for the design of finite impulse response digital filters

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Digital Signal Processing

- became increasingly relevant over the past 4 decades:

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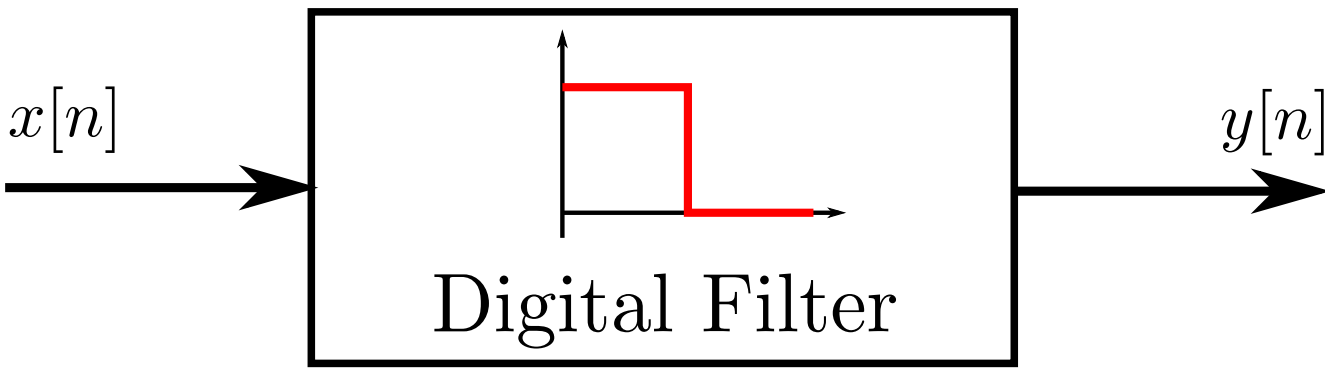
What are the 'engines' powering all these?

Digital filters



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$\rightarrow$ we get two categories of filters

- finite impulse response (**FIR**) filters
- infinite impulse response (**IIR**) filters

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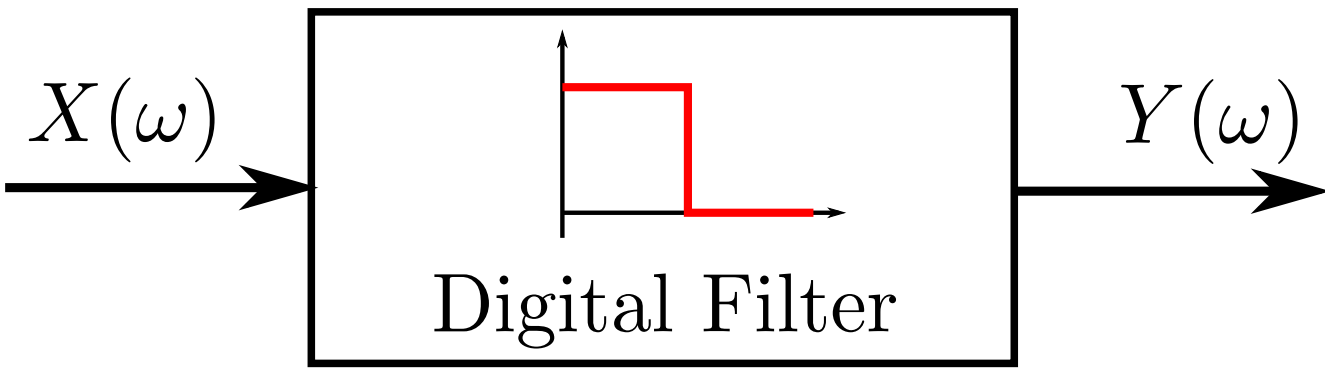

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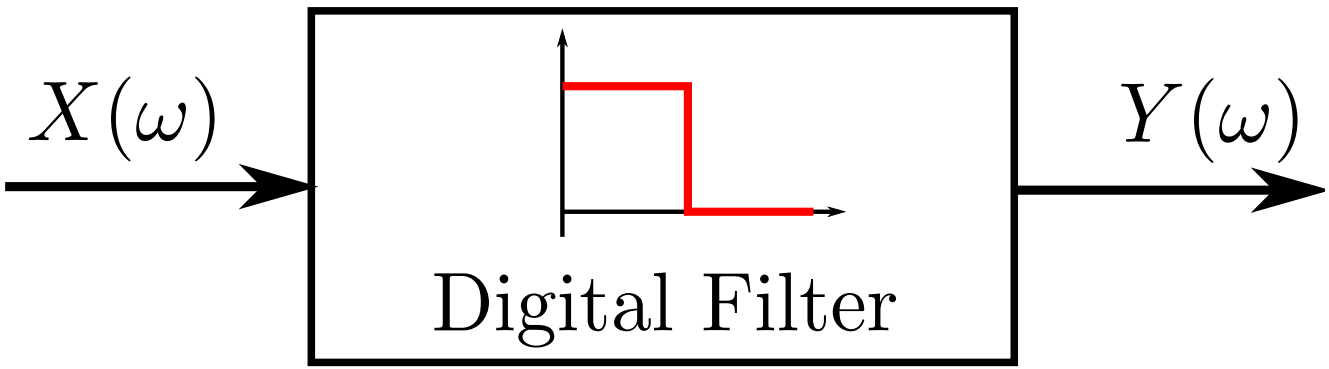

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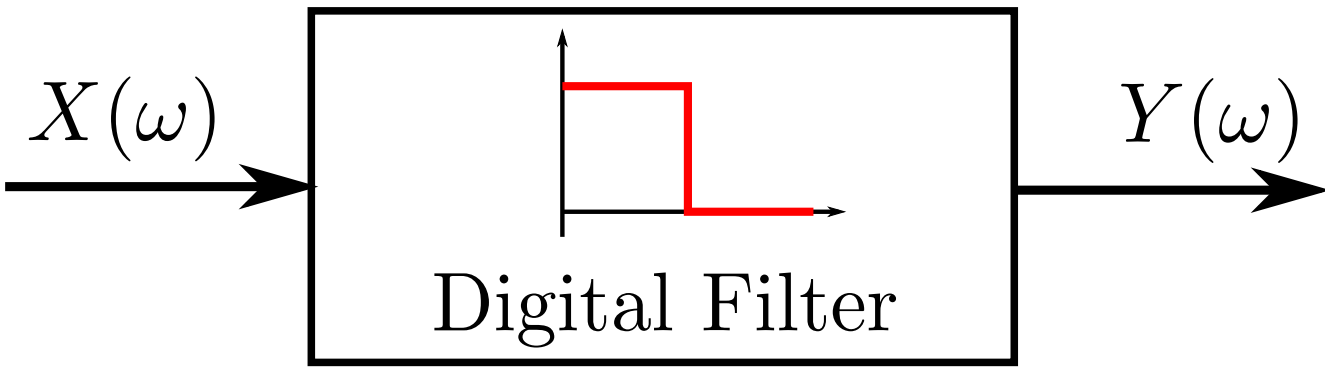

$$Y(\omega) = X(\omega)H(\omega), \omega \in [0, \pi]$$

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- finite impulse response (**FIR**) filters
  H is a polynomial
- infinite impulse response (**IIR**) filters
  H is a rational fraction

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The filtering toolchain

Steps:

1. **derive a concrete mathematical representation of the filter**
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Today's focus: **first step** for FIR filters

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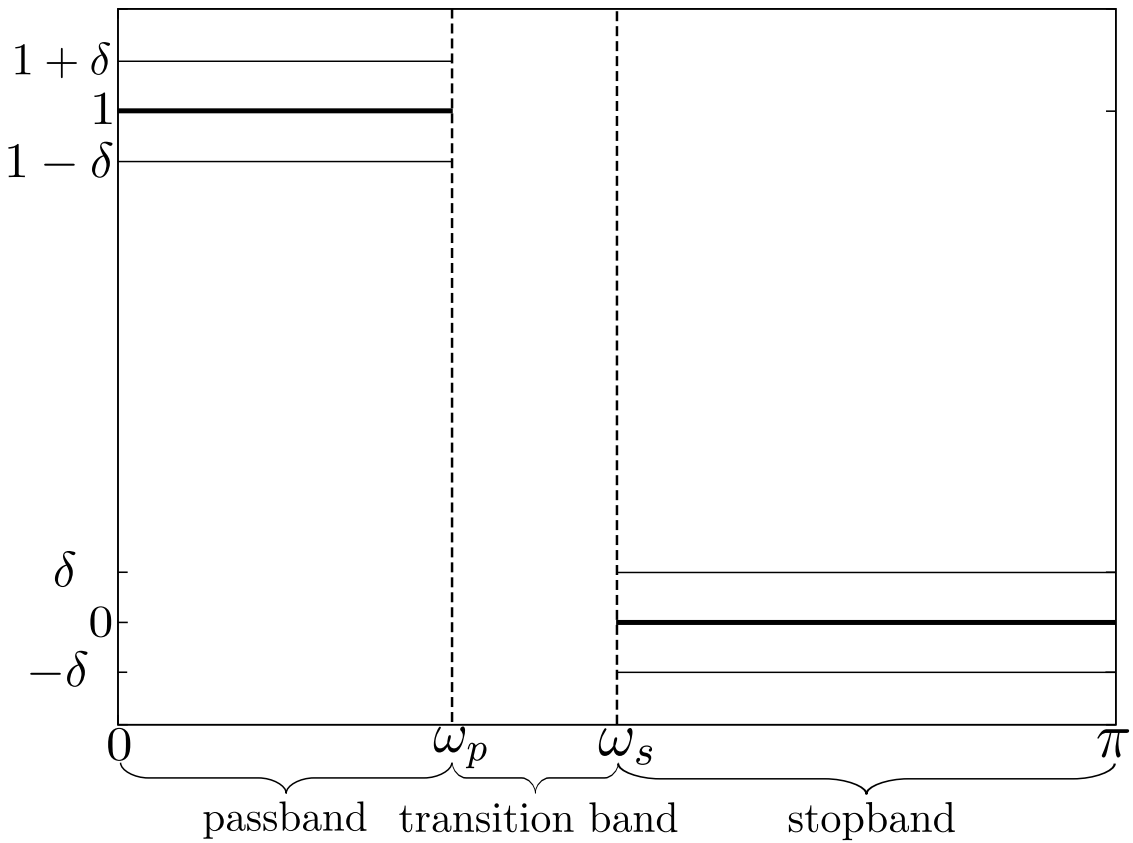

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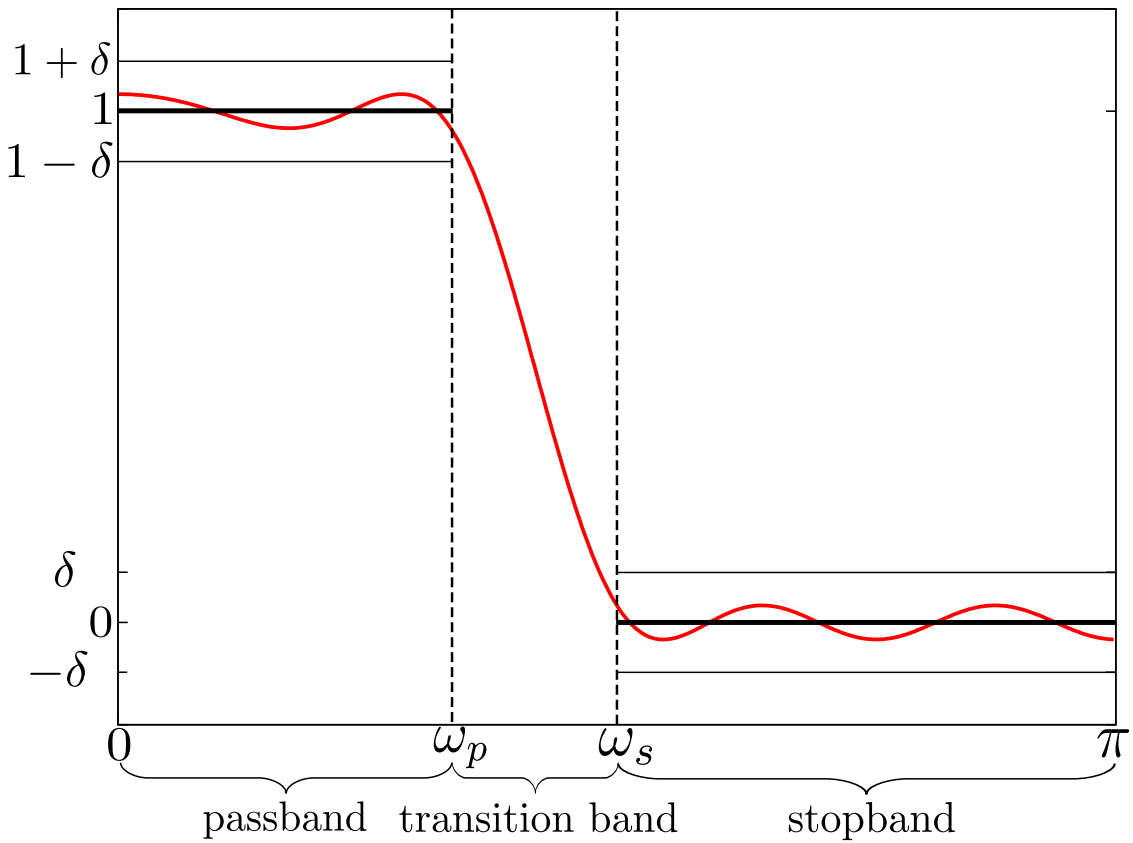


$$H(\omega) = \sum_{k=0}^{8} a_k \cos(\omega k)$$

Optimal FIR design with real coefficients

The problem: Given a closed real set F, find an approximation $H(\omega) = \sum_{k=0}^{L} a_k \cos(\omega k)$ of degree L for a continuous function $D(\omega), \omega \in F$ such that

$$\delta = \|E(\omega)\|_{\infty,F} = \max_{\omega \in F} |H(\omega) - D(\omega)|$$

is **minimal**.

Optimal FIR design with real coefficients

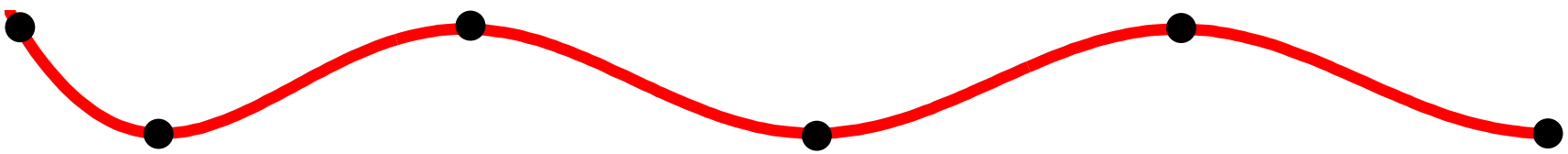
The solution: characterized by the **Alternation Theorem**

Theorem

*The **unique** solution* $H(\omega) = \sum_{k=0}^{L} a_k \cos(\omega k)$ *has an **error function*** $E(\omega)$*, for which there exist* $L+2$ *values* $\omega_0 < \omega_1 < \cdots < \omega_{L+1}$*, belonging to* F*, such that*

$$E(\omega_i) = -E(\omega_{i+1}) = \pm\delta,$$

for $i = 0, \ldots, L$.



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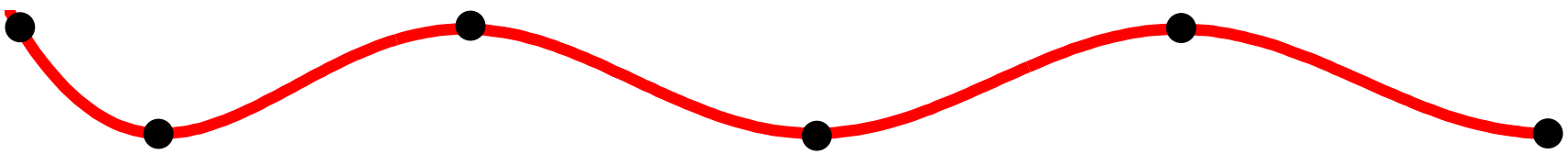


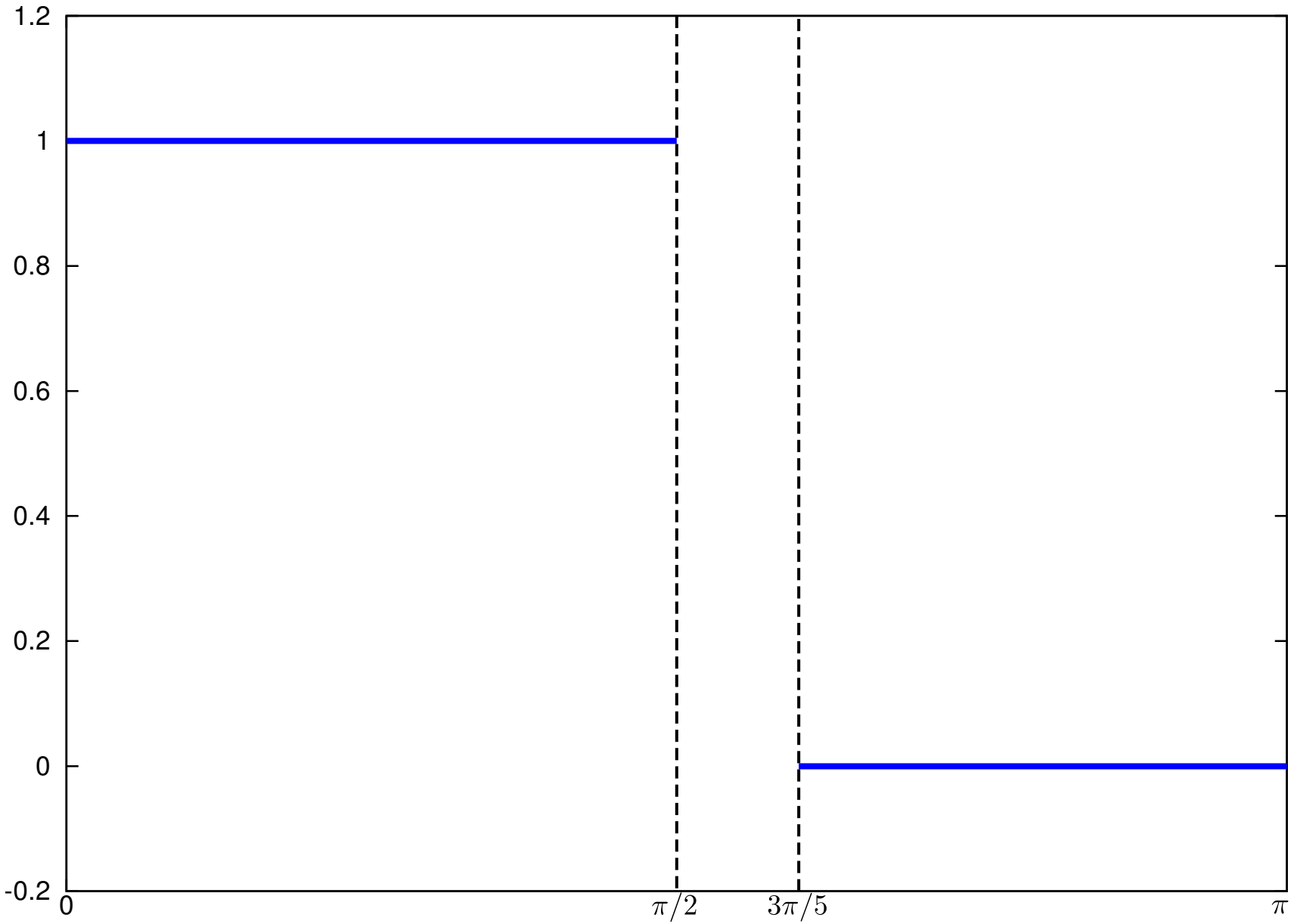
→ well studied in Digital Signal Processing literature

1972: Parks and McClellan

→ based on a powerful iterative approach from Approximation Theory:

1932: Remez

The Parks-McClellan design method: Example



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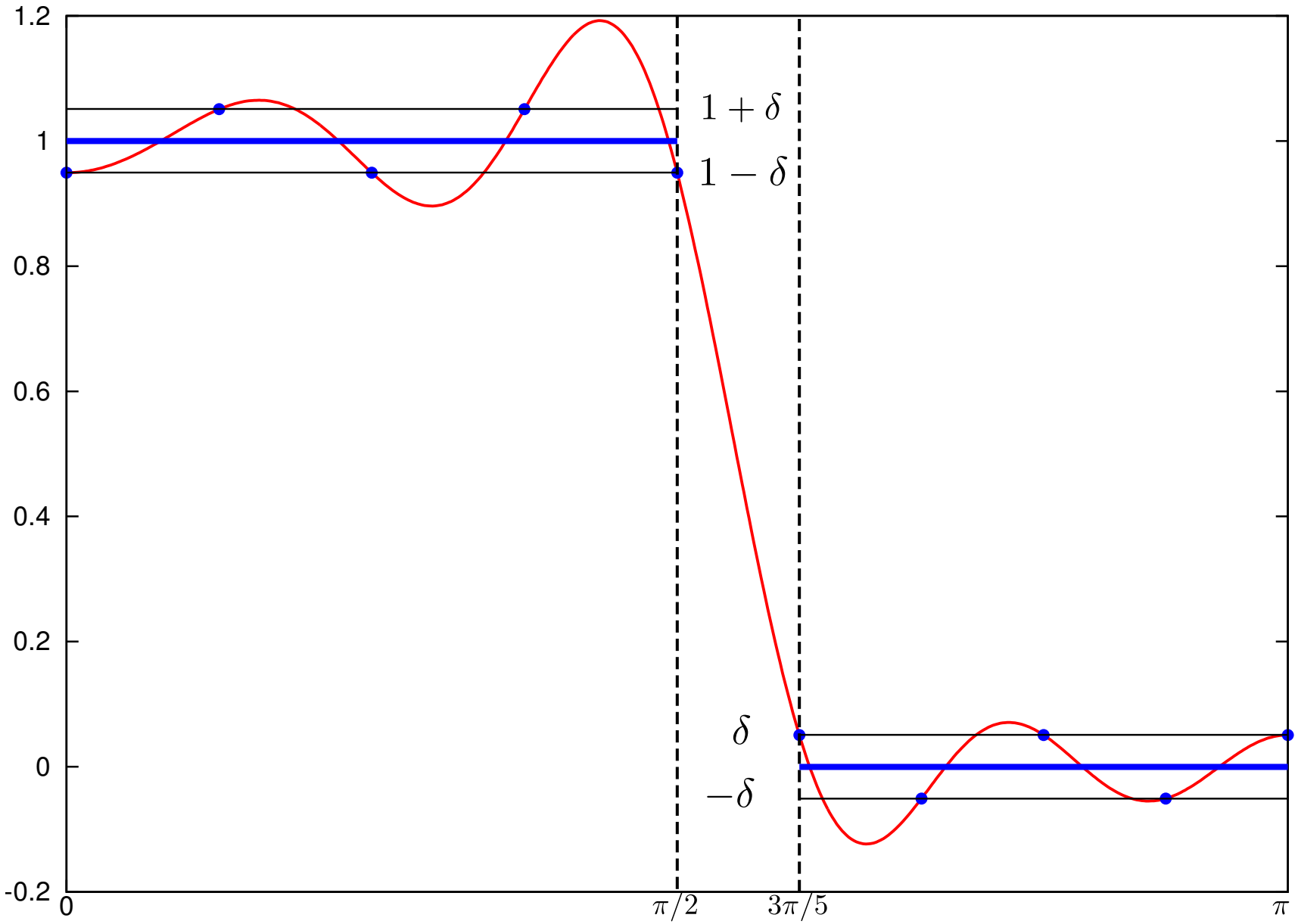

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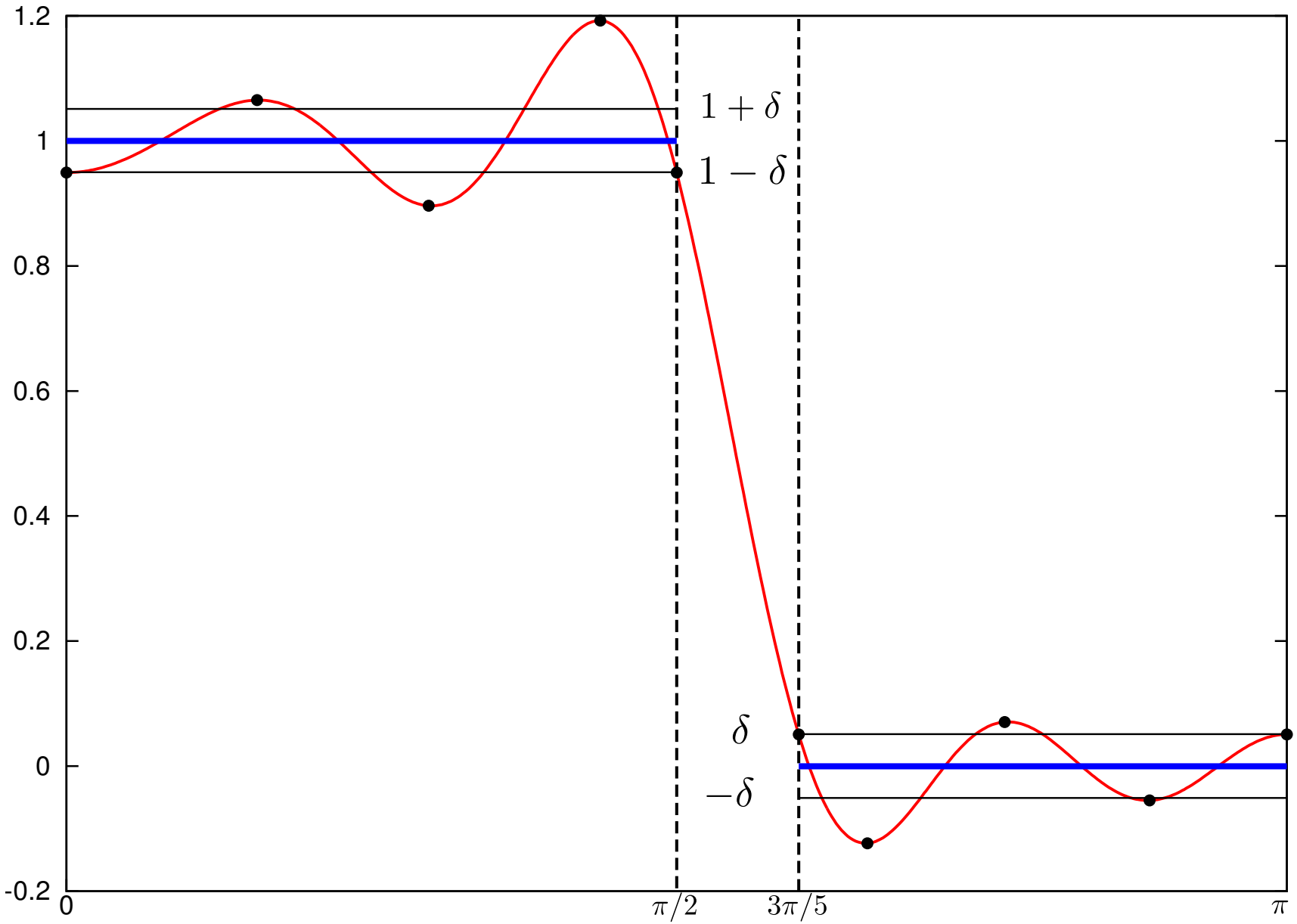

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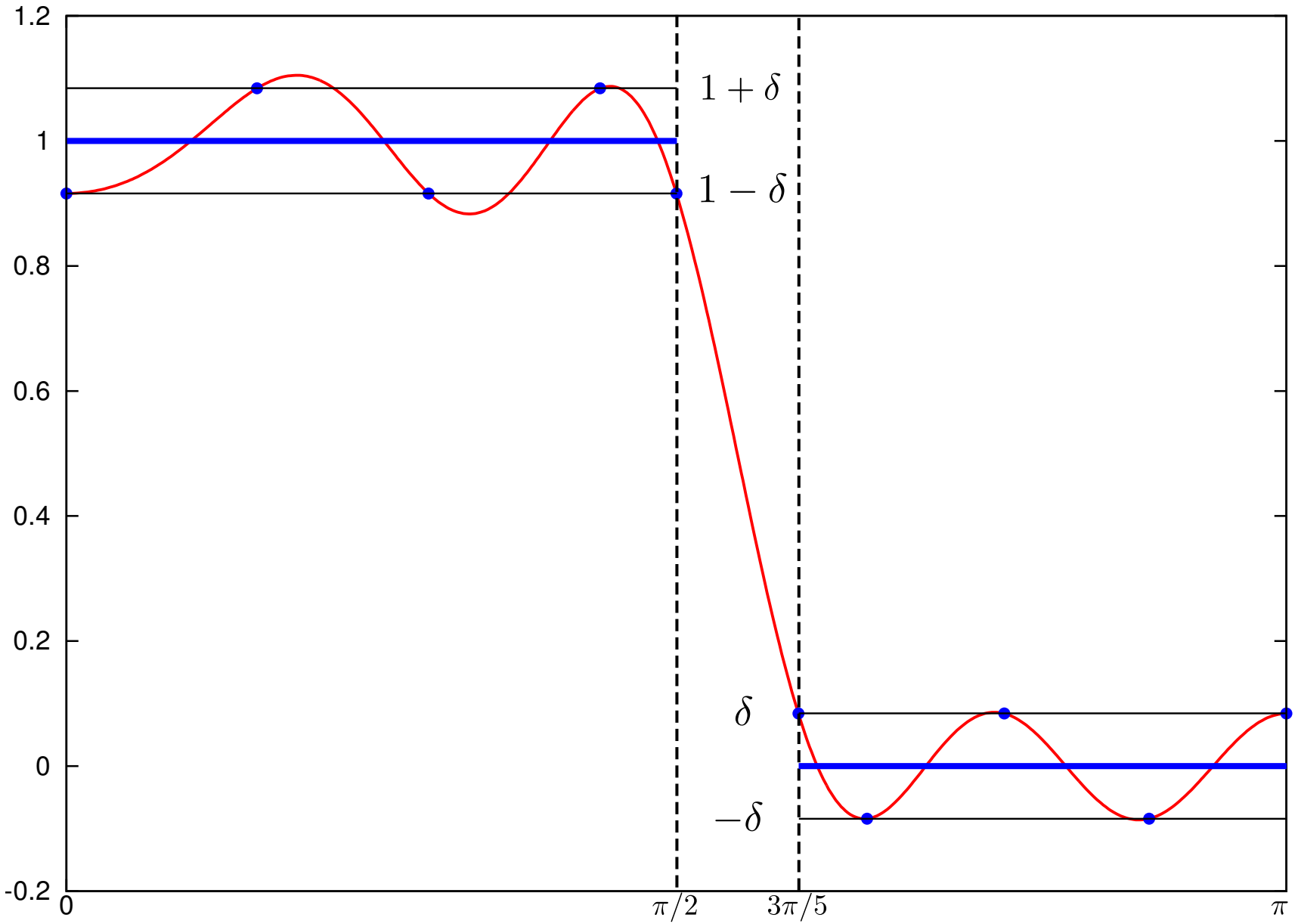

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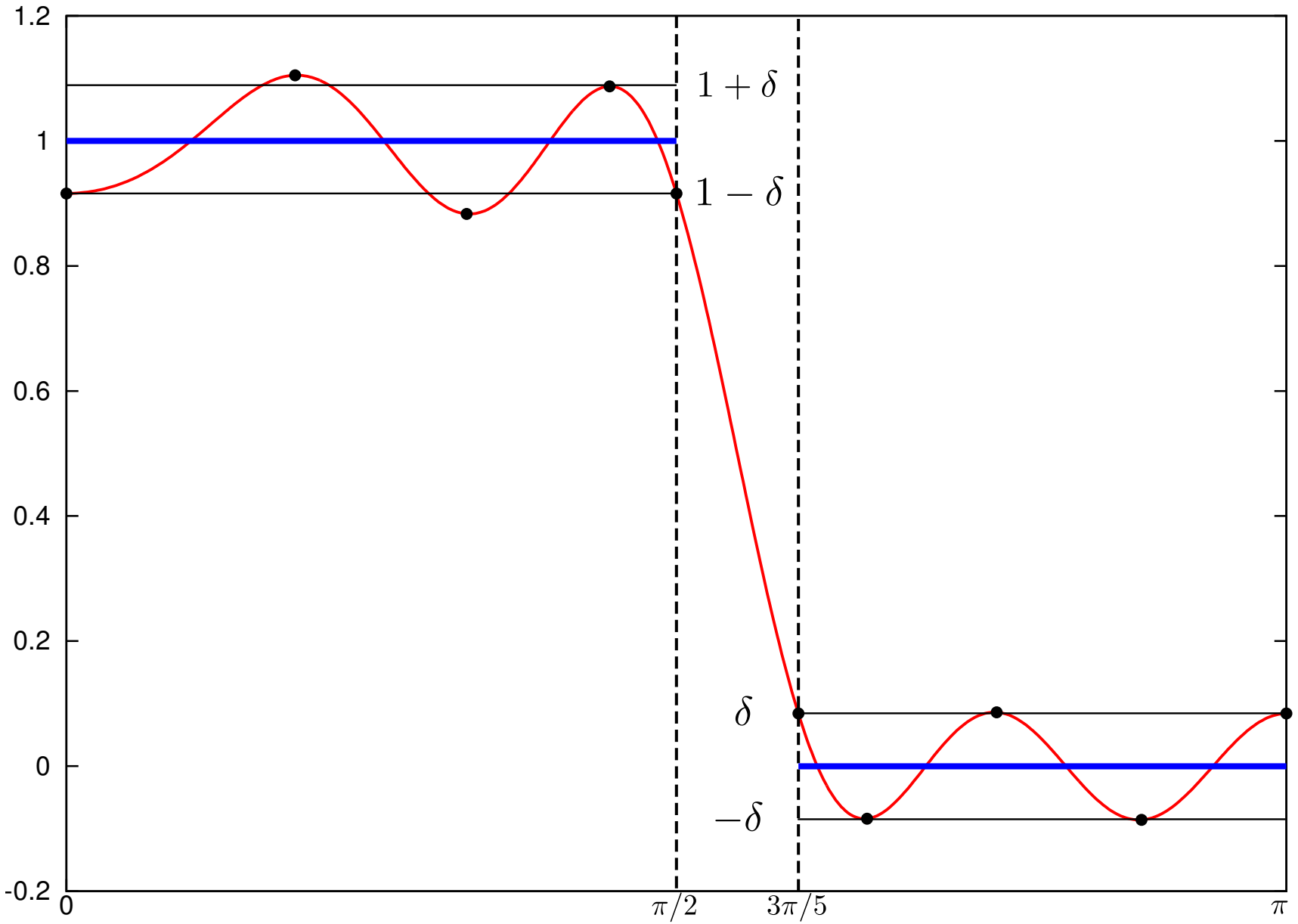

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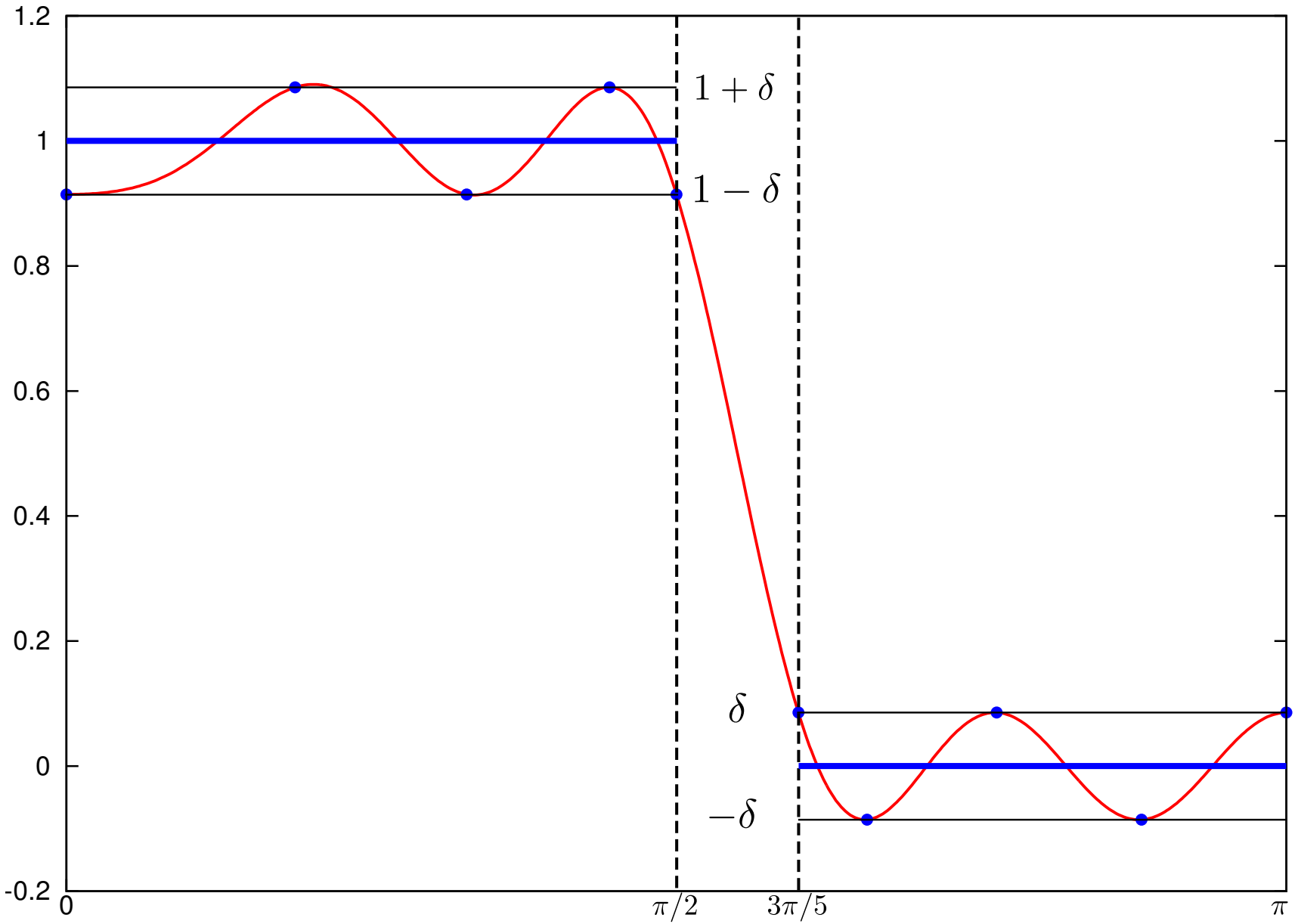

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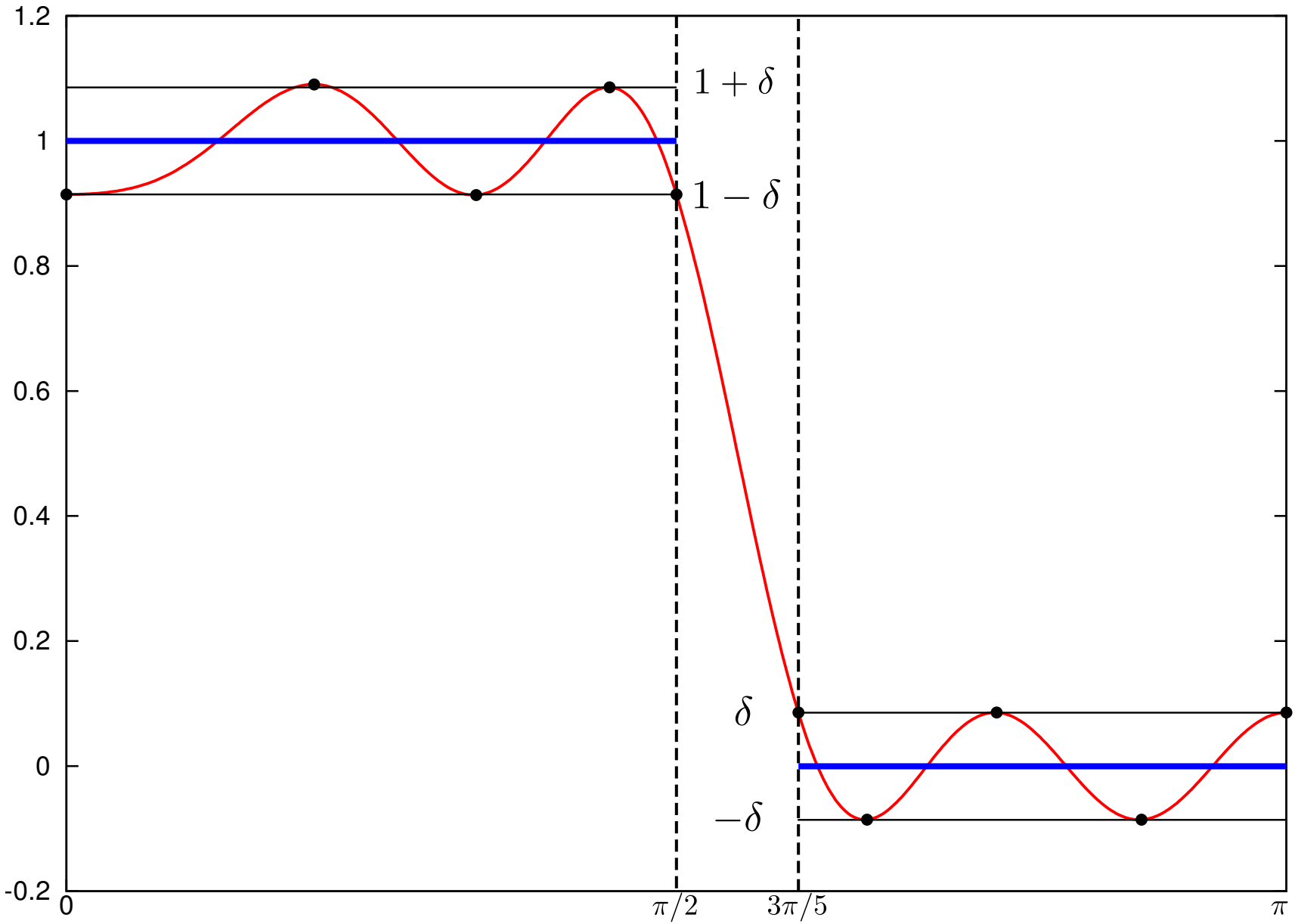

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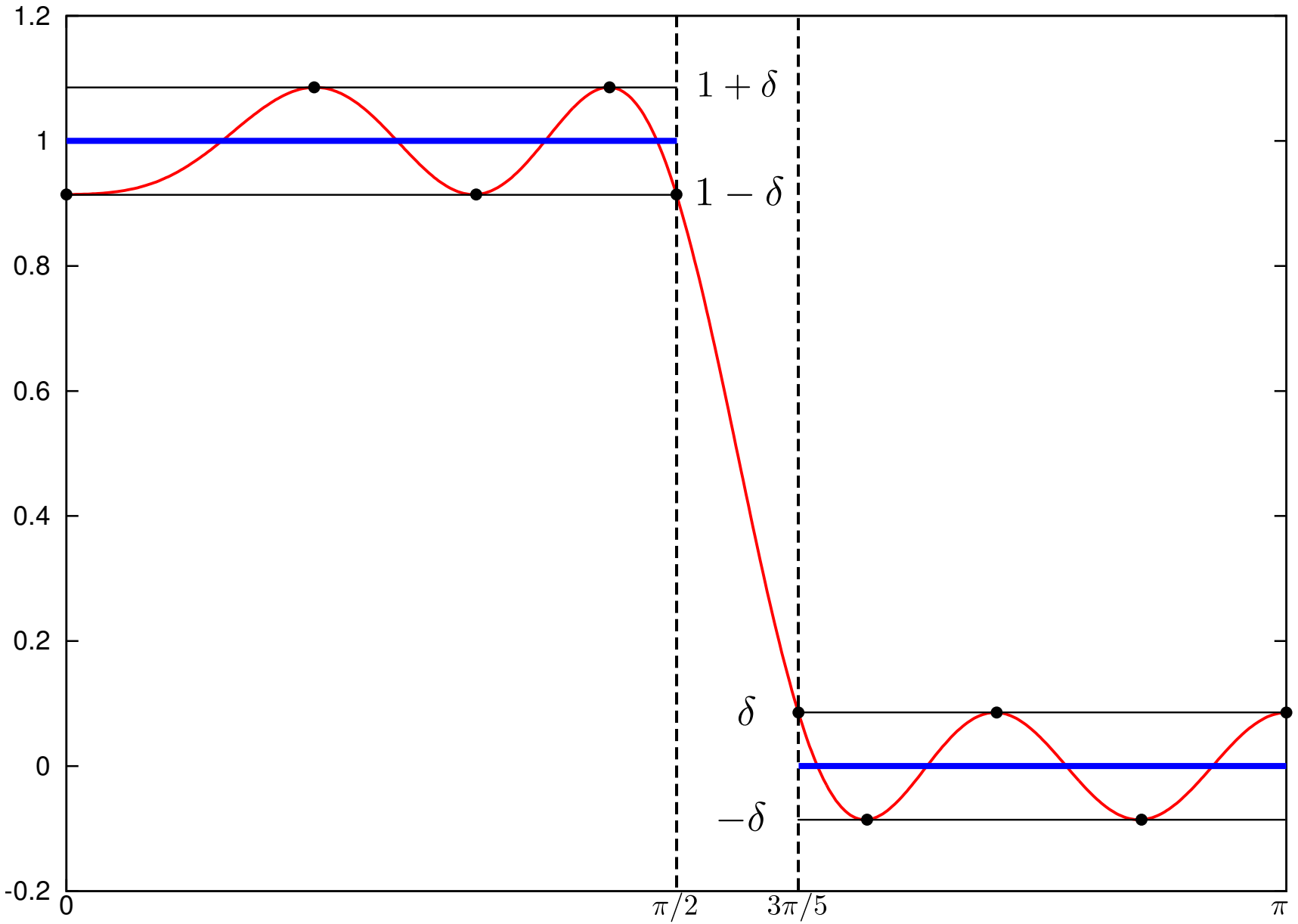
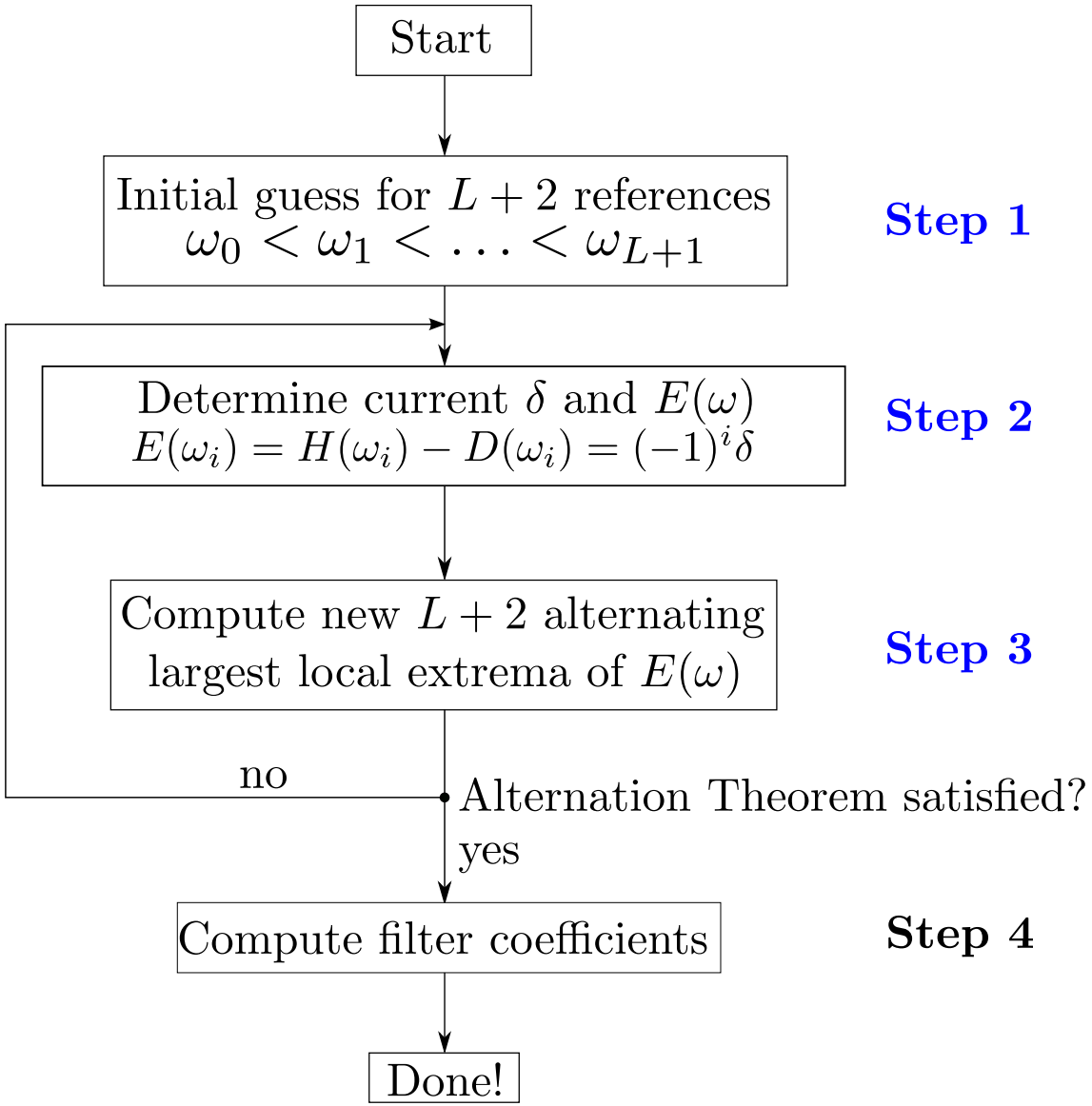


The Parks-McClellan design method: Steps



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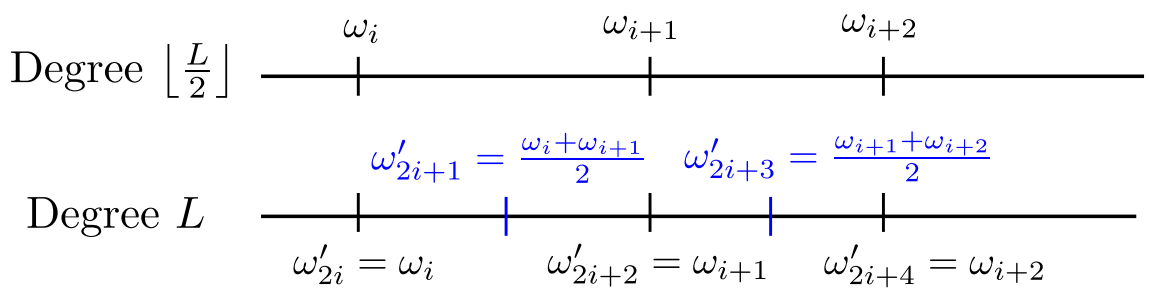
Our approach: extrema position extrapolation from smaller filters



$\rightarrow$ although empirical, it is rather robust in practice

Step 2: Computing the current error function $E(\omega)$ and δ

Amounts to solving a linear system in $a_0, \ldots, a_L$ and δ.

$$\begin{bmatrix} 1 & \cos(\omega_0) & \cdots & \cos(L\omega_0) & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \cos(\omega_L) & \cdots & \cos(L\omega_L) & (-1)^L \\ 1 & \cos(\omega_{L+1}) & \cdots & \cos(L\omega_{L+1}) & (-1)^{L+1} \end{bmatrix} \begin{bmatrix} a_0 \\ \vdots \\ a_L \\ \delta \end{bmatrix} = \begin{bmatrix} D(\omega_0) \\ \vdots \\ D(\omega_L) \\ D(\omega_{L+1}) \end{bmatrix}$$

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$\rightarrow$ use **barycentric** form of Lagrange interpolation [Berrut&Trefethen2004]

Barycentric Lagrange interpolation

Problem: p polynomial with $\deg p \leqslant L$ interpolates f at points x_j, i.e.,

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$\rightarrow$ the barycentric form of p is:

$$p(x) = \frac{\displaystyle\sum_{j=0}^{L} \frac{w_j}{x - x_j} f_j}{\displaystyle\sum_{j=0}^{L} \frac{w_j}{x - x_j}},$$

where $w_j = \dfrac{1}{\prod_{k \neq j}(x_j - x_k)}$.

Cost: $O(L^2)$ for computing *all* w_j, $O(L)$ for evaluating $p(x)$.

Barycentric Lagrange interpolation

$\rightarrow$ numerically stable **if the family of interpolation nodes used has a small Lebesgue constant** [Mascarenhas&Camargo2014]

The Lebesgue constant: specific to each grid of points; quantifies the convergence/divergence properties of polynomial interpolants using those nodes

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$\rightarrow$ from empirical observation, the families of points used inside the Parks-McClellan algorithm (Step 1 + Step 3) usually converge to sets of points with small Lebesgue constant

Step 3: Finding the local extrema of $E(\omega)$

Traditional approach: evaluate $E(\omega)$ on a dense grid of uniformly distributed points (in practice it is usually $16L$)
$\rightarrow$ works well for degree $L < 100$, tends to fail in some cases for larger L

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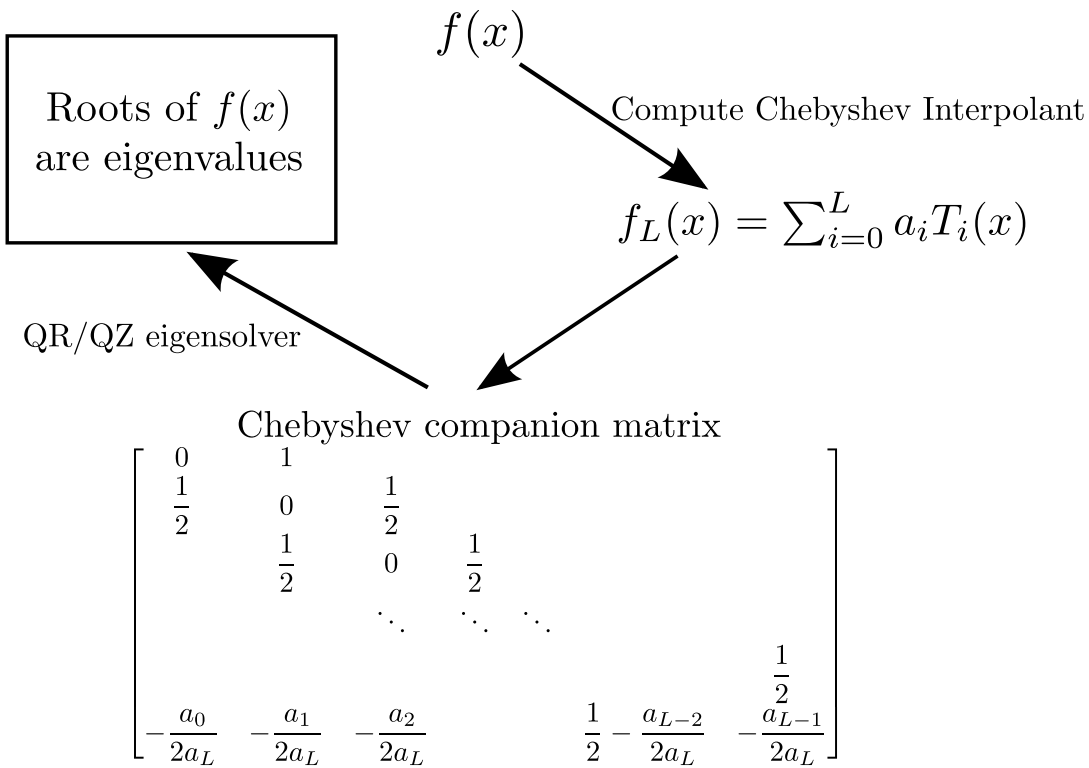
Our approach:

$\rightarrow$ **Chebyshev-proxy rootfinder(CPR)** [Boyd2002,Boyd2013]



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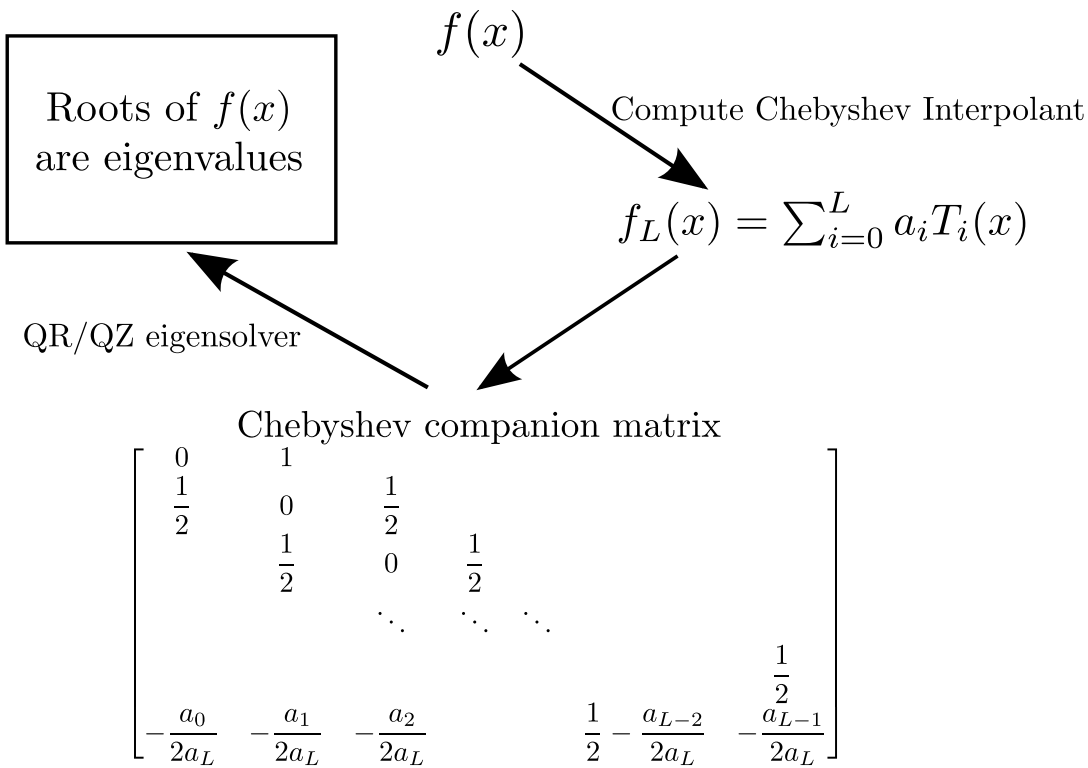

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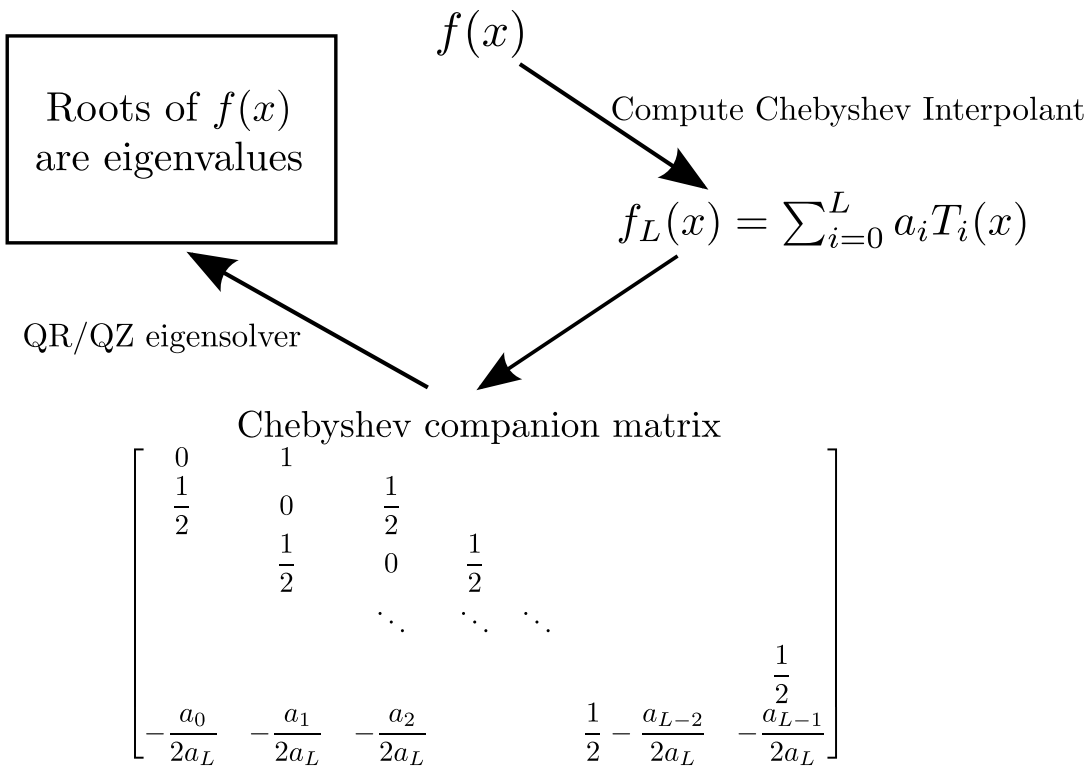

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$\rightarrow$ apply a similar idea for FIR approximations

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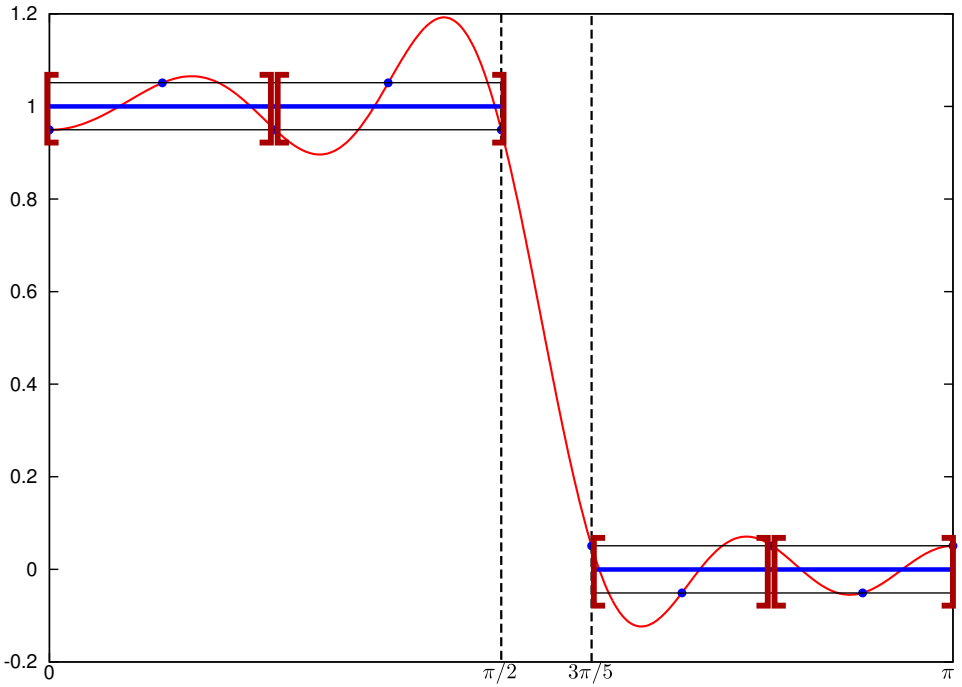

Idea:

- split F into *appropriate* subintervals $F = \bigcup_{i=0}^{N} F_i$
- interpolate $E(\omega)$ on each F_i with small degree Chebyshev interpolants C_i
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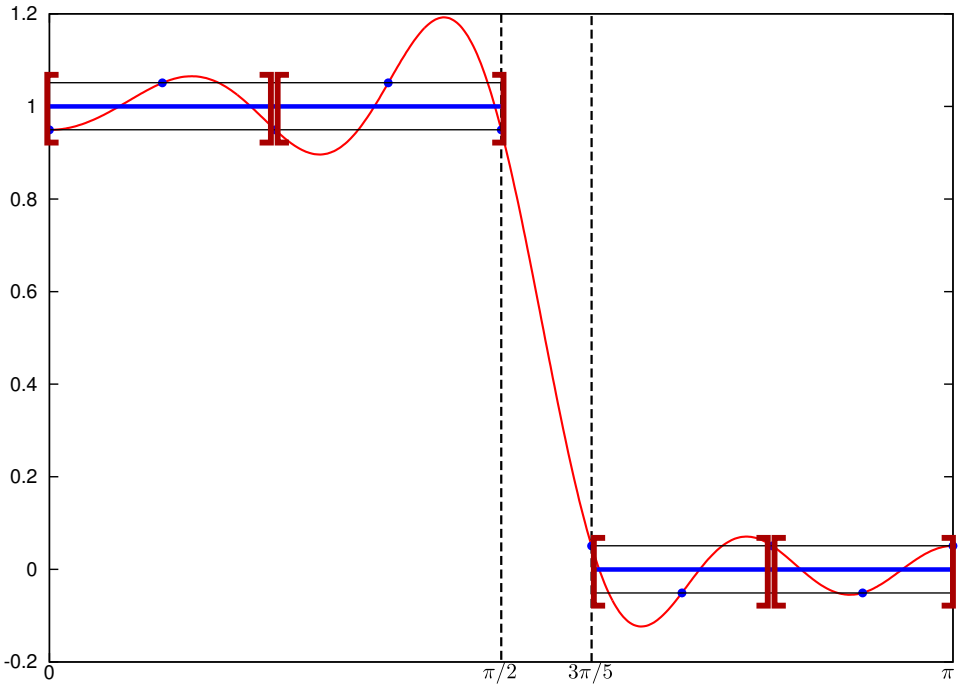

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Advantages:

$\rightarrow$ robust, numerically stable root finding approach

$\rightarrow$ easy to parallelize

Cost: if $N = O(L)$, overall $O(L^2)$ arithmetic operations

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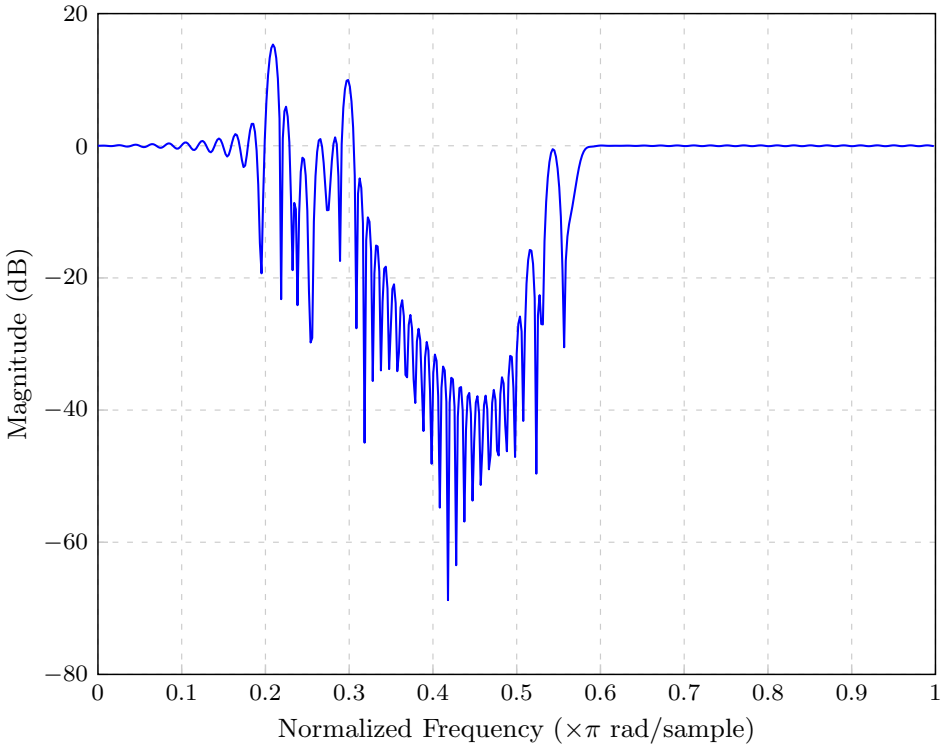
$\rightarrow$ can use the *Inverse Discrete Fourier Transform*

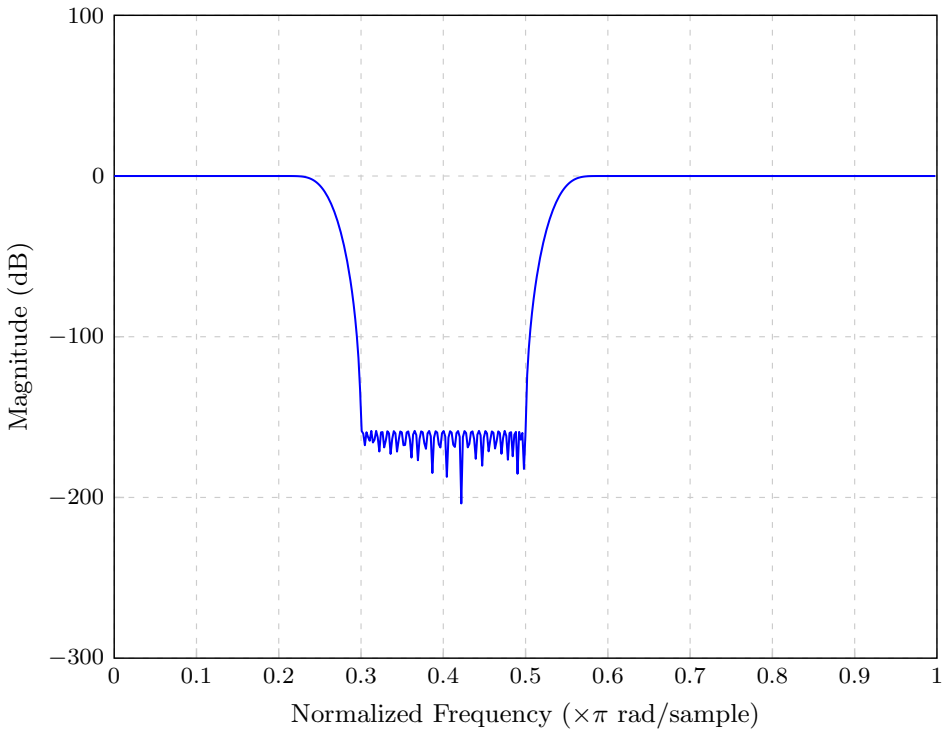
$\rightarrow$ implement it using Clenshaw's algorithm for computing linear combinations of Chebyshev polynomials (numerically robust approach)

Cost: $O(L^2)$ arithmetic operations

Our implementation: Examples & Results

Comparison with MATLAB & demo





Our implementation: Examples & Results

$\rightarrow$ design of channelizers for software radio systems

Specification: degree 53248 filter with stopband $\left[0, \frac{1}{8192}\pi\right]$ and passband $\left[\frac{3}{8192}\pi, \pi\right]$.

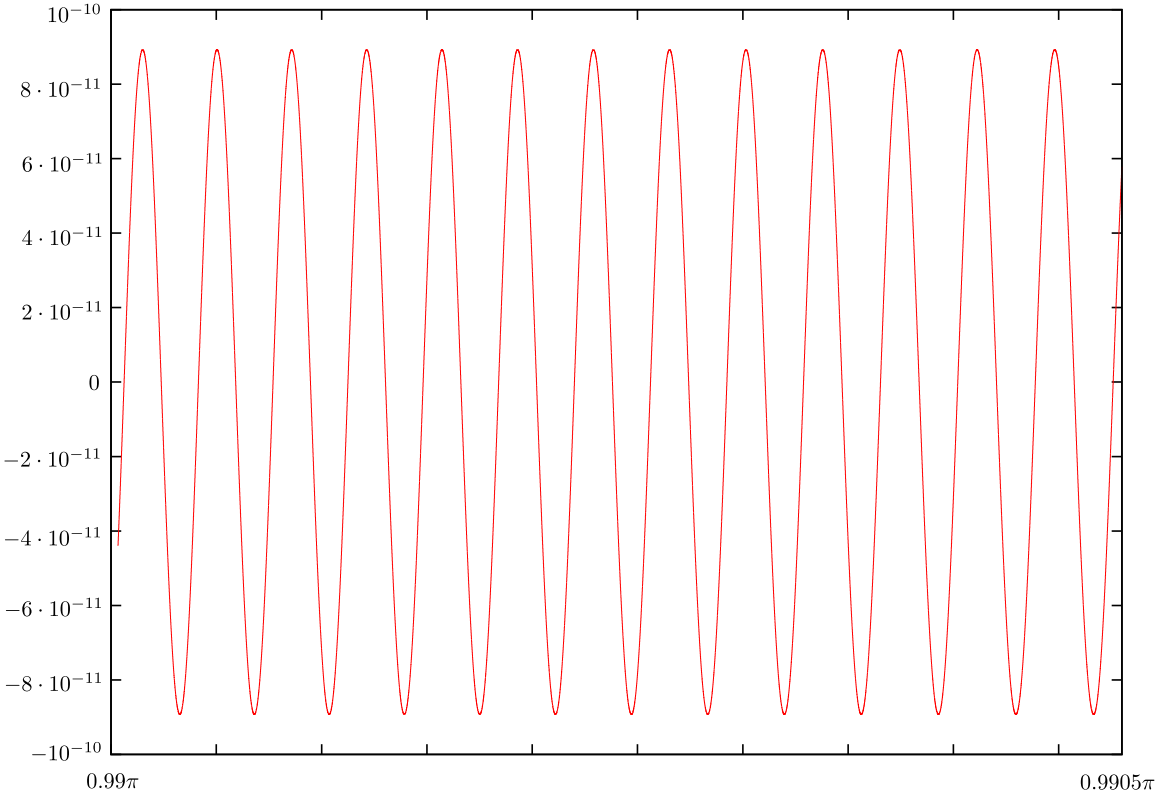


Figure: error function on $[0.99\pi, 0.9905\pi]$

Convergence: required 5 iterations

Perspectives

Conclusion:

- improved the practical behavior of a well known polynomial approximation algorithm for filter design
  $\rightarrow$ use numerically stable barycentric Lagrange interpolation + rootfinders without sacrifices in efficiency
- this new approach can take huge advantage of parallel architectures

Future work:

- release the code for our implementation as an open source library
- provide a complete toolchain for constructing FIR filters (approximation + quantification + hardware synthesis)
- tackle the IIR filter setting (rational fraction)
  - non-linear problem
  - constraints: poles located inside the unit circle