Exploring univariate mixed polynomials

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Univariate Mixed polynomial equation

- $P(z, \bar{z}) = 0$ of complex variable z, with complex or real coefficients.
- Equivalent to a pair of real bivariate polynomials $f(x, y) = 0$ and $g(x, y) = 0$.
- Specifying the degrees $n := deg_z P$, $m := deg_{\bar{z}} P$

$\rightarrow$ interesting roots structures and counting.

- A mix of complex and real algebra.

Applications:

- SVD of complex polynomial matrices.
- Complex moment problems.
- Harmonic maps.
- Real Milnor fibrations.

Examples

Writing $P = f(x, y) + ig(x, y)$,
the curves defined by $f = 0$ and $g = 0$ are shown in red and blue.
The roots in $\mathbb{C}$ are shown in green.

Example 1.

A random mixed polynomial of bidegree $(4, 1)$

$$P{:=}(4{-}3i)z^4\bar{z}{+}(3{+}7i)z^4{+}(8i)z^3\bar{z}{+}(7{+}9i)z^3{+}({-}6{-}9i)z^2\bar{z}$$

$$+(6{-}3i)z^2{+}({-}5{-}6i)z\bar{z}{+}(1{-}7i)z{+}({-}5{-}9i)\bar{z}{+}4{+}2i.$$

It has 3 roots.

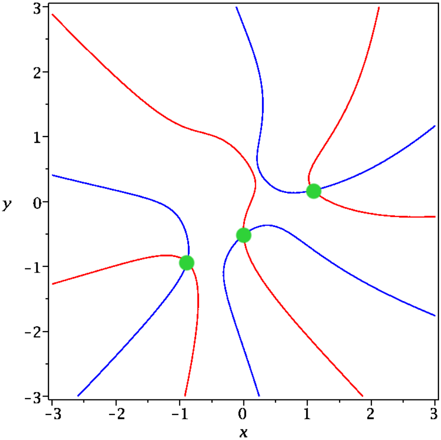

Figure: Example1

Example 2.

A mixed polynomial of bidegree $(8, 1)$ 17 roots.

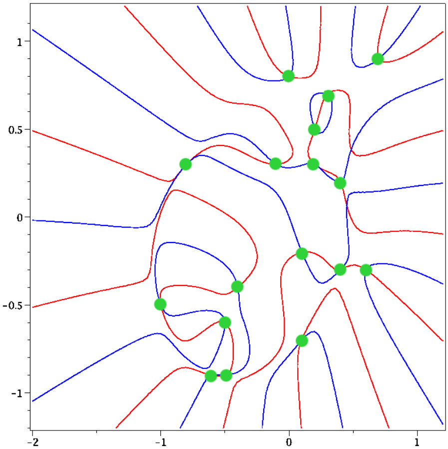

Figure: Example2

Example 3.

An example of bidegree $(4, 2)$ with real coefficients.
The non real roots appear by conjugated pairs.

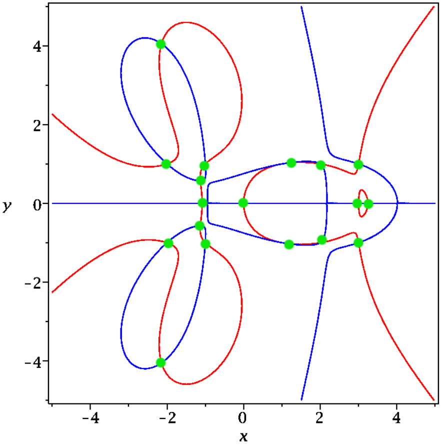

Figure: Example3

Example 4.

Examples of bidegree $(1, 1)$ with no punctual roots.

$$P = z\bar{z} + e$$

when $e = -1$, the roots form a circle;
while when $e = 1$, P has no root in $\mathbb{C}$.

Properties

Dimension

• The real variety $V(P)$ defined in $\mathbb{C} = \mathbb{R}^2$ by $P = 0$, where P is an univariate mixed polynomial (non identically zero), can be either of dimension 1, 0 or -1 (i.e. $V(P)$ is empty).

Lemma

The only possible curve contained in the zero set $V(P)$ of a mixed polynomial $P(z, \bar{z})$ of bidegree $(n, 1)$ is either a circle or a line.

Algebra

- Factorization properties and algebraic algorithms, valid for bivariate polynomials, are also valid for univariate mixed polynomials.

- Fast interpolation can be adapted in that setting.

Vandermonde

- One can construct “natural” Vandermonde matrices adapted to mixed polynomials, for simple or multiple roots.

Proposition

The Vandermonde matrices corresponding to suited number of distinct simple points is invertible.
Similarly for double roots.

- We used them to compute examples, optimizing on a reduced set of parameteres.

Topological degree

• Let $P = f + ig$ be a (generic) mixed polynomial of bidegree (n, m).
At each of the N single isolated root $z_j = (x_j, y_j)$, $j = 1..N$ of P, the local topological degree of $(f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ at (x_j, y_j) is the sign of its jacobian determinant.

• A circle containing all the roots, can be viewed as a circle "around infinity".

• So, $N = (n - m) + 2K$, where K is the number of roots with negative jacobian; there, the map (f, g) is localy attractive.

Resultant

- A complex roots of $P(z, \bar{z})$ is also a root of $\bar{P}(w, z)$, such that $w = \bar{z}$.

Lemma

A complex root of $P(z, \bar{z})$ is also a root of $R(z)$, the "biprojectif" resultant of $P(z, w)$ and $\bar{P}(w, z)$;
which has degree at most $n^2 + m^2$.
So we get a first bound on the number of complex roots of P.

Mixed polynomials of bidegree $(n, 1)$

• The study of this important case reduces to a "classical" subject: Counting roots of a rational harmonic map $\bar{z} = r(z)$, with $r(z) := \frac{p(z)}{q(z)}$.

• This equation provides a simple but effective modelization of "gravitational lensing":
Following Einstein relativity theory, the light from a star is deviated by the presence of other stars. The observation can be studied on a plane projection identified with the complex plane.

Theorem[Khavinson,Neumann 05]
The number $N(r,n)$ of roots of $\bar{z} = r(z)$ is bounded by $5n-5$.

Theorem[Rhie 03]
There exists a family of rational fractions $r_n,\ n > 1$ such that $N(r_n, n) = 5n-5$.

Theorem[Bleher, Homma, Ji, Roeder 12]
There exists a family of rational fractions $r_{n,k},\ n > 1, k = 0, .., 2n-2$, such that $N(r_{n,k}, n) = n - 1 + 2k$.

Physical (and other) configurations

- $r(z) = \frac{p}{q} := \sum \frac{\mu_j}{z - z_j}$,

where $gcd(q, q') = 1,\ gcd(p, q) = 1$; μ_j are positive masses.

- Rhie's example is a small deformation of a regular configuration of identical stars around a circle and near its origin.
- We explored, thanks to Vandermonde interpolation and also by considering random data, other examples of configurations.

When $n = 5$, we have $2n + 1 = 11$, $n^2 + 1 = 26$, $5n - 5 = 20$.

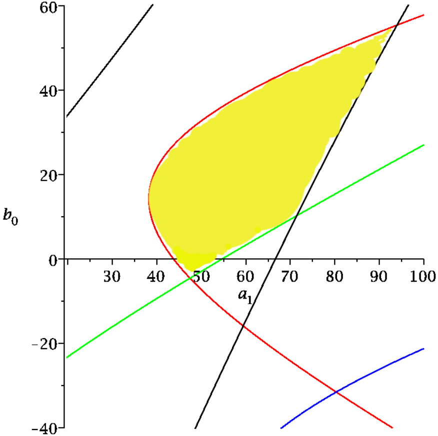

Figure: 4 conics in the parameters space

• We interpolated at the origin and at 4 pairs of conjugated roots, Then, we constructed 4 conics defined by the jacobians of (f, g) at these 4 pairs.

• We delimited a region where 3 of the 4 jacobians were negative.

• We chose in that region the values $a_0 = 70, b_0 = 40$ which corresponds to a mixed polynomial with 8 pairs of conjugated roots and 4 real roots.

• The 8 attractive fixed points of $\bar{z} = r(z)$ are indicated by black solid boxes, and the 12 repelling ones by green solid discs,

• the 5 poles of r are indicated by brown diamonds.

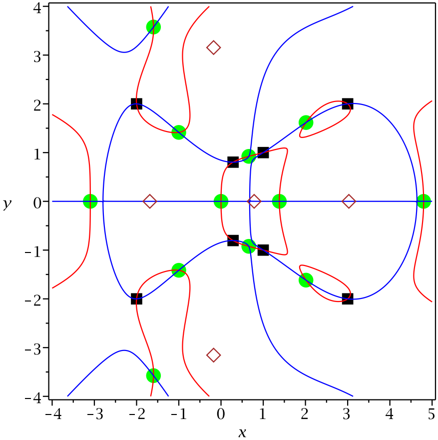

Figure: 20 roots

Random Mixed polynomials

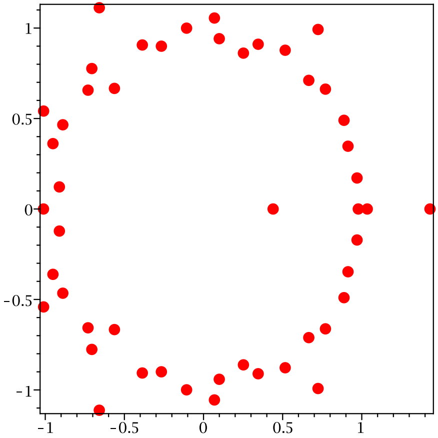

Figure: Equation with uniform coefficients

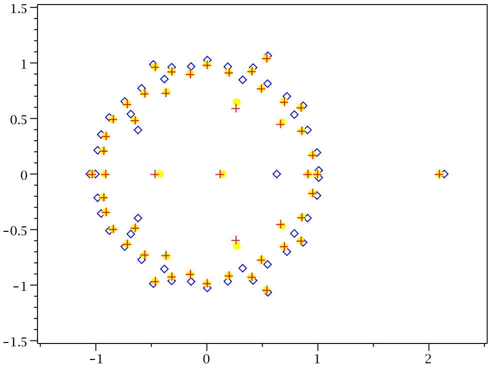

Figure: Equation with equidistributed poles

Conclusion

• We presented univariate mixed polynomials from a Computer algebra view point, and recorded application in gravitational lensing.

• Little is known on roots of mixed equations of bidegrees (n, m) with small m.

So, the next step is exploring in detail the equation $\bar{z}^2 = r(z)$.