A new method to compute the probability of collision for short-term space encounters

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Fiction...

Credit Gravity (2013)

On-orbit collision

Figure: Cerise hit by a debris in 1996 (source : CNES/D. Ducros)

On-orbit collision

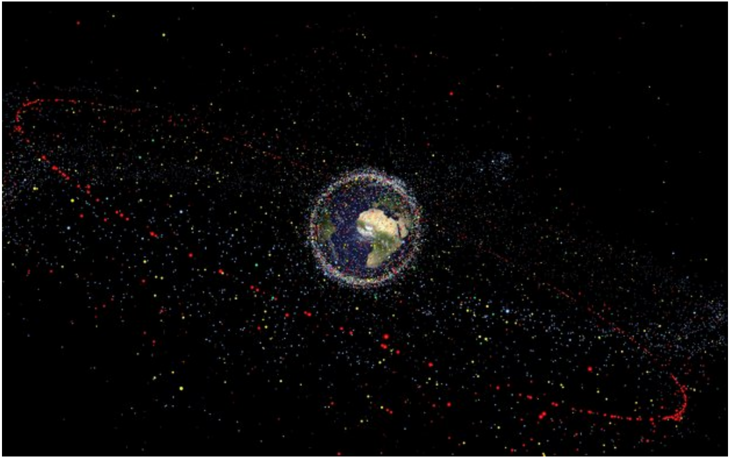

Figure: Space debris population model (source : ESA)

On-orbit collision

Context

- Two objects: primary P (operational satellite) and secondary S (space debris)
- Information about their geometry, position, velocity at a given time
  $\rightsquigarrow$ Affected by uncertainty
- Needs:
  - Risk assessment
  - Design of a collision avoidance strategy

  $\rightsquigarrow$ compute the probability of collision

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Classical assumptions

- Spherical objects
- Gaussian probability density functions
- Independent probability distribution laws

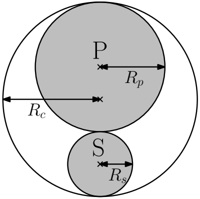

Figure: Combined spherical object

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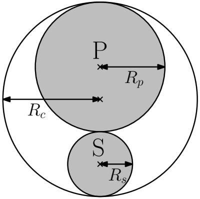

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Probability of collision

Generally: 12-dimensional, Gaussian integrand, Complex integration domain

Computation: Monte-Carlo trials and/or simplified models

Short-term encounter model and probability of collision

- Framework: High relative velocity
- Assumptions:
  - Rectilinear relative motion
  - No velocity uncertainty
  - Infinite encounter time horizon

$\rightsquigarrow$ Probability of collision:
2-D integral over a disk.

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Formula

$$\mathcal{P} = \frac{1}{2\pi\sigma_x\sigma_y} \int_{\mathcal{B}((0,0),R)} \exp\left(-\frac{(x-x_m)^2}{2{\sigma_x}^2} - \frac{(y-y_m)^2}{2{\sigma_y}^2}\right) \mathrm{d}x\mathrm{d}y,$$

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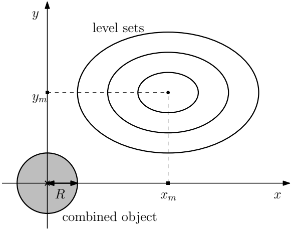

Figure: 2-D Gaussian integral over a disk

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where

- R: radius of combined object
- x_m, y_m: mean relative coordinates
- σ_x, σ_y: standard deviations of relative coordinates

Existing methods

- Methods based on numerical integration schemes:
  Foster '92, Patera '01, Alfano '05.
- Analytic methods: Chan '97 $\rightsquigarrow$ uses some simplifying assumptions ($\sigma_x = \sigma_y$)

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  $\rightsquigarrow$ provides truncation error bounds

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Our purpose

Give a "simple", "analytic" formula, suitable for double-precision evaluation
and effective error bounds.

Our method - Underlying techniques

1. Laplace transform:
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   $\rightsquigarrow$ solution of linear differential equation with polynomial coefficients
   $\rightsquigarrow$ power series coefficients satisfy a linear recurrence relation with polynomial coefficients

Example:

$$f(x) = \exp(x) \quad \leftrightarrow \quad \{f' - f = 0,\ f(0) = 1\} \leftrightarrow \{(n+1)f_{n+1} = f_n, f_0 = 1\}$$

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Sketch of the proof - Laplace Transform

$\forall z \in \mathbb{R}^+$:

$$g(z) := \mathcal{P}(\sqrt{z}) = \frac{1}{2\pi\sigma_x\sigma_y} \int_{\mathcal{B}((0,0),\sqrt{z})} \exp\left(-\frac{(x-x_m)^2}{2{\sigma_x}^2} - \frac{(y-y_m)^2}{2{\sigma_y}^2}\right) \mathrm{d}x\mathrm{d}y, \tag{1}$$

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$\mathcal{L}(g)$ is D-finite !

Sketch of the proof - Borel-Laplace

$$\begin{tikzcd}
g(z) \arrow[r, "\text{Laplace Transform}"] \arrow[d] & \mathcal{L}(g)(t) = \dfrac{\exp\left(-\frac{{\sigma_x}^2 {y_m}^2 + {\sigma_y}^2 {x_m}^2}{2{\sigma_x}^2{\sigma_y}^2} + \frac{{y_m}^2}{2{\sigma_y}^2(2t{\sigma_y}^2+1)} + \frac{{x_m}^2}{2{\sigma_x}^2(2t{\sigma_x}^2+1)}\right)}{t\sqrt{(2t{\sigma_x}^2+1)(2t{\sigma_y}^2+1)}} \arrow[d, "\text{expansion at } \infty"] \\
g(z) = \sum_{i=0}^{\infty} \frac{l_i}{(i+1)!} z^{i+1} & \hat{\mathcal{L}}(g)(t) := t^2 \mathcal{L}(g)\left(\frac{1}{t}\right) = \sum_{i=0}^{\infty} l_i \left(\frac{1}{t}\right)^i \arrow[l, "\text{Borel Transform}"']
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$g(z)$ is:

- D-finite
- entire function of exponential type
- type $\sigma = \frac{1}{2\sigma_y^2}$

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Cancellation in finite precision power series evaluation

Example: $\sigma_x = 115, \sigma_y = 1.41, x_m = 0.15, y_m = 3.88, \sqrt{z} = 15$

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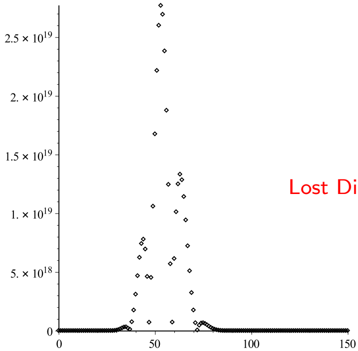

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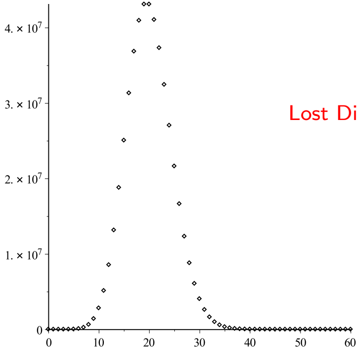

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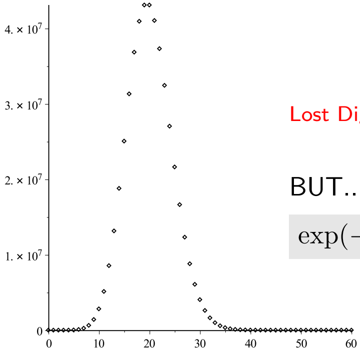

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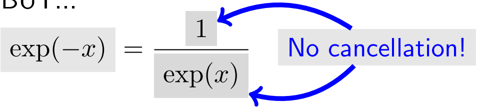

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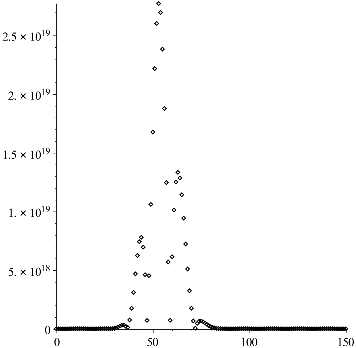

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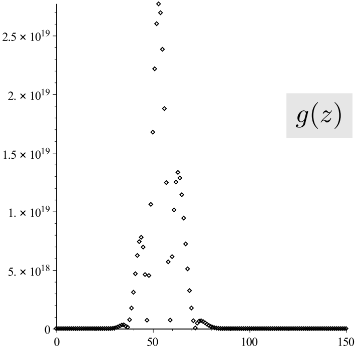

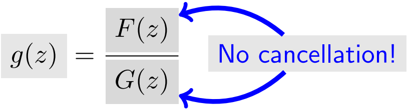

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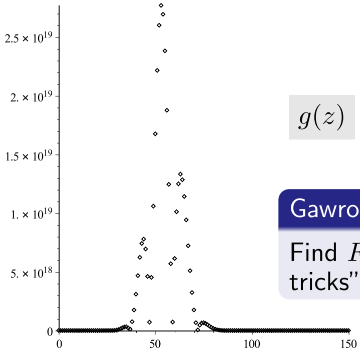

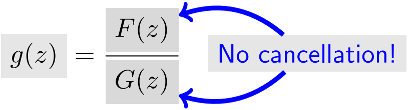

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Gawronski, Müller, Reinhard (2007) Method:

Find $F(z)$ and $G(z)$ using "complex analysis tricks" $\rightsquigarrow$ indicator function.

Sketch of the proof - Borel-Laplace + GMR Method

$$\begin{tikzcd}
g(z)=\frac{F(z)}{G(z)} \arrow[r, "\text{Laplace}"'] & \mathcal{L}(g)(t)=\frac{\exp\left(-\frac{{\sigma_x}^2{y_m}^2+{\sigma_y}^2{x_m}^2}{2{\sigma_x}^2{\sigma_y}^2}+\frac{{y_m}^2}{2{\sigma_y}^2(2t{\sigma_y}^2+1)}+\frac{{x_m}^2}{2{\sigma_x}^2(2t{\sigma_x}^2+1)}\right)}{t\sqrt{(2t{\sigma_x}^2+1)(2t{\sigma_y}^2+1)}} \arrow[d, "\text{expansion at } \infty"] \\
g(z)=\sum_{i=0}^{\infty}\frac{l_i}{(i+1)!}z^{i+1} \arrow[u] & \hat{\mathcal{L}}(g)(t):=t^2\mathcal{L}(g)\left(\frac{1}{t}\right)=\sum_{i=0}^{\infty}l_i\left(\frac{1}{t}\right)^i \arrow[l, "\text{Borel Transform}"']
\end{tikzcd}$$

$g(z)$ is:

- D-finite
- entire function of exponential type
- type $\sigma=\frac{1}{2\sigma_y^2}$

$\hat{\mathcal{L}}(g)(t)$ is:

- D-finite
- Finite radius of convergence $2\sigma_y^2$
- l_i - P-recursive

Sketch of the proof - Borel-Laplace + GMR Method

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\end{tikzcd}$$

$g(z)$ is:

- D-finite
- entire function of exponential type
- type $\sigma = \frac{1}{2\sigma_y^2}$
- indicator function:
  $\left|g(re^{i\theta})\right| \sim \exp\left(h(\theta)r\right)$ for large r

$$h(\theta) = \begin{cases} \frac{-\cos\theta}{2{\sigma_y}^2} & \text{if } \theta \in [\frac{\pi}{2}, \frac{3\pi}{2}] \\ 0 & \text{if } \theta \in [0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi]. \end{cases}$$

$\hat{\mathcal{L}}(g)(t)$ is:

- D-finite
- Finite radius of convergence $2\sigma_y^2$
- l_i - P-recursive

Zoom on Indicator functions

$$g(z) = \frac{F(z)}{G(z)}$$

- $\left|g(re^{i\theta})\right| \sim \exp\left(h(\theta)r\right)$ for large r
- indicator of g:

$$h(\theta) = \begin{cases} \frac{-\cos\theta}{2{\sigma_y}^2} & \text{if } \theta \in [\frac{\pi}{2}, \frac{3\pi}{2}] \\ 0 & \text{otherwise.} \end{cases}$$

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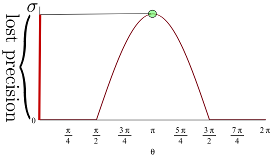

lost precision $\sim |\sigma - h(0)|$

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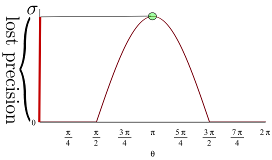
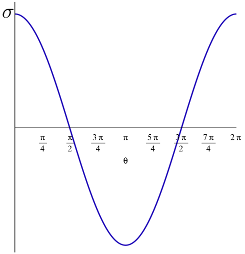
- $G(z) = \exp(\sigma z)$
  indicator of G:



lost precision $\sim |\sigma - h(0)|$

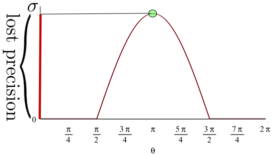


Zoom on Indicator functions

$G(z)g(z) = F(z)$

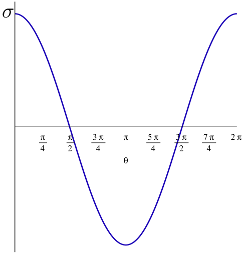
- $\left|g(re^{i\theta})\right| \sim \exp\left(h(\theta)r\right)$ for large r
- indicator of g:

$$h(\theta) = \begin{cases} \frac{-\cos\theta}{2{\sigma_y}^2} & \text{if } \theta \in [\frac{\pi}{2}, \frac{3\pi}{2}] \\ 0 & \text{otherwise.} \end{cases}$$

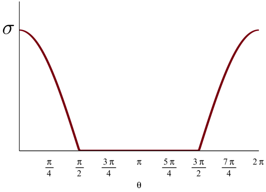


lost precision $\sim |\sigma - h(0)|$

- $G(z) = \exp(\sigma z)$
  indicator of G:



- indicator of F:



Sketch of the proof - Borel-Laplace + GMR Method

$$\begin{tikzcd}
G(z)g(z) = F(z) \arrow[r, "\text{Laplace}"'] & \mathcal{L}(g)(t-\sigma) \arrow[dr, "\text{expansion at } \infty"] & \\
G(z)g(z) = \sum\limits_{k=0}^{\infty} \frac{\alpha_k}{(k+1)!} z^{k+1} \arrow[u] & & \sum\limits_{k=0}^{\infty} \alpha_k \left(\frac{1}{t}\right)^k \arrow[ll, "\text{Borel Transform}"']
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- $G(z) = \exp(\sigma z)$
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- reduced cancellation for evaluating F, G on positive real line
- recurrence for α_k:

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\end{tikzcd}$$

- $G(z) = \exp(\sigma z)$
- F is D-finite
- reduced cancellation for evaluating F, G on positive real line
- recurrence for α_k:

$$\begin{aligned}
-(32k\sigma_x^4\sigma_y^{10} + 128\sigma_x^4\sigma_y^{10})\alpha_{k+4} &= (\sigma_x^4 y_m^2 - 2\sigma_x^2\sigma_y^2 y_m^2 + \sigma_y^4 y_m^2)\alpha_k \\
&+((-4\sigma_x^4\sigma_y^4 + 8\sigma_x^2\sigma_y^6 - 4\sigma_y^8)k - 10\sigma_x^4\sigma_y^4 - 6\sigma_x^4\sigma_y^2 y_m^2 + 20\sigma_x^2\sigma_y^6 + 8\sigma_x^2\sigma_y^4 y_m^2 - 10\sigma_y^8 - 2\sigma_y^6 y_m^2)\alpha_{k+1} \\
&+((24\sigma_x^4\sigma_y^6 - 32\sigma_x^2\sigma_y^8 + 8\sigma_y^{10})k + 72\sigma_x^4\sigma_y^6 + 12\sigma_x^4\sigma_y^4 y_m^2 - 92\sigma_x^2\sigma_y^8 - 8\sigma_x^2\sigma_y^6 y_m^2 + 20\sigma_y^{10} + 4\sigma_y^8 x_m^2)\alpha_{k+2} \\
&+((-48\sigma_x^4\sigma_y^8 + 32\sigma_x^2\sigma_y^{10})k - 168\sigma_x^4\sigma_y^8 - 8\sigma_x^4\sigma_y^6 y_m^2 + 104\sigma_x^2\sigma_y^{10} - 8\sigma_y^{10} x_m^2)\alpha_{k+3},
\end{aligned}$$

Prop. Coefficients α_k are positive

Suppose $\sigma_x > \sigma_y$.

$$\hat{L}(x) = \sum_{k=0}^{\infty} \alpha_k x^k \text{ satisfies } \hat{L}'(x) = F(x)\hat{L}(x), \quad \hat{L}(0) = \frac{\exp\left(-\frac{\sigma_x^2 y_m^2 + \sigma_y^2 x_m^2}{2\sigma_x^2\sigma_y^2}\right)}{2\sigma_x\sigma_y}, \text{ where}$$

$$F(x) = \frac{{y_m}^2}{4{\sigma_y}^4} + \frac{{\sigma_y}^4{x_m}^2}{\left(x\,\left({\sigma_x}^2 - {\sigma_y}^2\right) - 2\,{\sigma_x}^2{\sigma_y}^2\right)^2} - \frac{1}{-2\,{\sigma_y}^2 + x} + \frac{-{\sigma_x}^2 + {\sigma_y}^2}{2x\,\left({\sigma_x}^2 - {\sigma_y}^2\right) - 4\,{\sigma_x}^2{\sigma_y}^2}.$$

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$$F(x) = \frac{{y_m}^2}{4{\sigma_y}^4} + \frac{{\sigma_y}^4{x_m}^2}{\left(x\,\left({\sigma_x}^2 - {\sigma_y}^2\right) - 2\,{\sigma_x}^2{\sigma_y}^2\right)^2} - \frac{1}{-2\,{\sigma_y}^2 + x} + \frac{-{\sigma_x}^2 + {\sigma_y}^2}{2x\,\left({\sigma_x}^2 - {\sigma_y}^2\right) - 4\,{\sigma_x}^2{\sigma_y}^2}.$$

$$f_k = \frac{1 + \dfrac{\left(1 - \dfrac{\sigma_y^2}{\sigma_x^2}\right)^k \left((k+1)\left(\dfrac{x_m^2\sigma_y^2}{\sigma_x^4}\right) + 1 - \dfrac{\sigma_y^2}{\sigma_x^2}\right)}{2}}{(2\sigma_y^2)^{k+1}} + \begin{cases} 0, & k > 0 \\ \dfrac{{y_m}^2}{4\sigma_y^4}, & k = 0, \end{cases}$$

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$$0 \le f_k = \frac{1 + \dfrac{\left(1 - \dfrac{\sigma_y^2}{\sigma_x^2}\right)^k \left((k+1)\left(\dfrac{x_m^2\sigma_y^2}{\sigma_x^4}\right) + 1 - \dfrac{\sigma_y^2}{\sigma_x^2}\right)}{2}}{(2\sigma_y^2)^{k+1}} + \begin{cases} 0, & k > 0 \\ \dfrac{{y_m}^2}{4\sigma_y^4}, & k = 0, \end{cases}$$

$$0 \le (n+1)\alpha_{n+1} = \sum_{i=0}^{n} f_i \cdot \alpha_{n-i}, \rightsquigarrow \text{by induction.}$$

Bounds using majorant series

Let $\gamma := \dfrac{1 + \frac{1}{2}\left(1 - \frac{\sigma_y^2}{\sigma_x^2} + \frac{x_m^2\sigma_y^2}{\sigma_x^4} + \frac{{y_m}^2}{{\sigma_y}^2}\right)}{2{\sigma_y}^2}$, $\overline{\alpha}_k := \alpha_0\gamma^k$ and $\underline{\alpha}_k := \alpha_0\left(\dfrac{1}{2{\sigma_y}^2}\right)^k$.

Then $\underline{\alpha}_k \leq \alpha_k \leq \overline{\alpha}_k, \forall k \in \mathbb{N}$.

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Then $\underline{\alpha}_k \leq \alpha_k \leq \overline{\alpha}_k, \forall k \in \mathbb{N}$.

Let $\tilde{P}_N(z) := \sum_{k=0}^{N-1} \dfrac{\alpha_k z^k}{(k+1)!}$. Then we have the following error bounds:

$$\underline{\varepsilon}_N(z) \leq g(z) - \tilde{P}_N(z) \leq \overline{\varepsilon}_N(z), \text{ where}$$

$$\underline{\varepsilon}_N(z) := 2\alpha_0 \sigma_y^2 e^{-\frac{z}{2\sigma_y^2}} \frac{\left(\frac{z}{2\sigma_y^2}\right)^{N+1}}{(N+1)!},$$

$$\overline{\varepsilon}_N(z) := \frac{\alpha_0}{\gamma} e^{-\frac{z}{2\sigma_y^2}} \frac{(z\gamma)^{N+1}}{N+1!}.$$

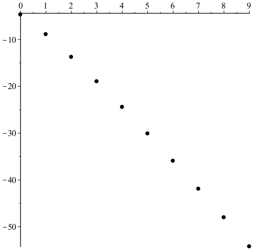
Examples

Sample 1

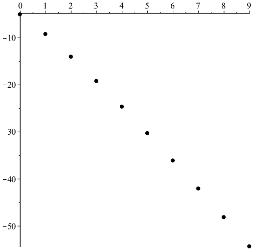
Case #	Input parameters (km)				
	σ_x	σ_y	R	x_m	y_m
1	0.05	0.025	0.005	0.01	0
2	0.05	0.025	0.005	0	0.01
3	0.075	0.025	0.005	0.01	0
4	0.075	0.025	0.005	0	0.01
5	3	1	0.01	1	0
6	3	1	0.01	0	1
7	3	1	0.01	10	0
8	3	1	0.01	0	10
9	10	1	0.01	10	0
10	10	1	0.01	0	10
11	3	1	0.05	5	0
12	3	1	0.05	0	5

Examples: quality $\eta = \dfrac{\overline{\varepsilon}_{10}(z) - \underline{\varepsilon}_{10}(z)}{\underline{\varepsilon}_{10}(z) + \tilde{P}_{10}(z)}$ and plot $\log\left(\tilde{p}_i z^i\right)$

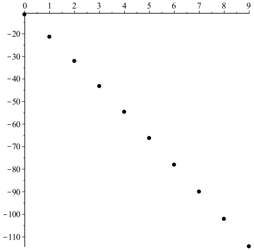
Case 1: $\eta = 23$



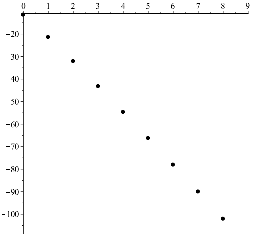
Case 2: $\eta = 22$



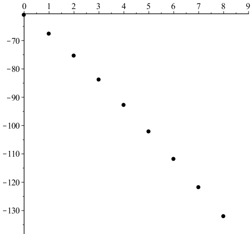
Case 4: $\eta = 22$



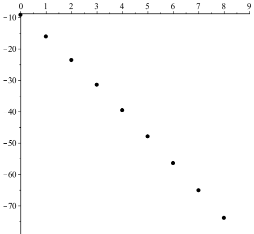
Case 6: $\eta = 47$



Case 8: $\eta = 33$



Case 11: $\eta = 34$



Numerical study

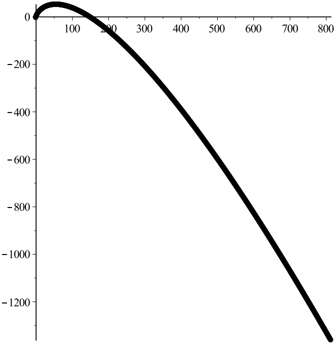
Sample 1

Case	Probability of Collision (-)			
#	Alfano	Patera	Chan	New method
1	9.742×10^{-3}	9.741×10^{-3}	9.754×10^{-3}	9.742×10^{-3}
2	9.181×10^{-3}	9.181×10^{-3}	9.189×10^{-3}	9.181×10^{-3}
3	6.571×10^{-3}	6.571×10^{-3}	6.586×10^{-3}	6.571×10^{-3}
4	6.125×10^{-3}	6.125×10^{-3}	6.135×10^{-3}	6.125×10^{-3}
5	1.577×10^{-5}	1.577×10^{-5}	1.577×10^{-5}	1.577×10^{-5}
6	1.011×10^{-5}	1.011×10^{-5}	1.011×10^{-5}	1.011×10^{-5}
7	6.443×10^{-8}	6.443×10^{-8}	6.443×10^{-8}	6.443×10^{-8}
8	0	3.219×10^{-27}	3.216×10^{-27}	3.219×10^{-27}
9	3.033×10^{-6}	3.033×10^{-6}	3.033×10^{-6}	3.033×10^{-6}
10	0	9.656×10^{-28}	9.645×10^{-28}	9.656×10^{-28}
11	1.039×10^{-4}	1.039×10^{-4}	1.039×10^{-4}	1.039×10^{-4}
12	1.564×10^{-9}	1.564×10^{-9}	1.556×10^{-9}	1.564×10^{-9}

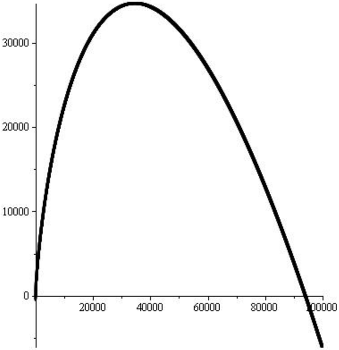
— equal to reference value (from [Chan 2008])

Examples: quality $\eta_N = \dfrac{\overline{\varepsilon}_N(z) - \underline{\varepsilon}_N(z)}{\underline{\varepsilon}_N(z) + \tilde{P}_N(z)}$ and plot $\log\left(\tilde{p}_i z^i\right)$

Case 3 Alfano: $\eta_{800} = 30$



Case 5 Alfano: $\eta_{121000} = 20$



Sample 2 (from [Alfano 2009])

Case #	Input parameters (m)				
	σ_x	σ_y	R	x_m	y_m
3	114.25	1.41	15	0.15	3.88
5	177.8	0.038	10	2.12	-1.22

Examples

Sample 2 (from [Alfano 2009])

Case #	Input parameters (m)				
	σ_x	σ_y	R	x_m	y_m
3	114.25	1.41	15	0.15	3.88
5	177.8	0.038	10	2.12	-1.22

Alfano's test case number	Probability of collision (-)		
	Alfano	New method	Reference (MC)
3	0.10038	0.10038	0.10085
5	0.044712	0.045509	0.044499

Conclusion

New method

- Analytical formula
- Reduced cancellation evaluation
- Error bounds
- No simplifying assumption

Current and future work

- Saddle-point method for "degenerate" cases
- Long-term 3D encounter model
- Extension to polygonal cross-sections
- Extensive testings and comparisons with existing methods