Computing real points on determinantal varieties and spectrahedra

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Luminy — 4/11

Linear matrices and Spectrahedra

A linear matrix is a polynomial matrix of degree 1:

$$\mathsf{A}(x) = \mathsf{A}_0 + x_1\mathsf{A}_1 + \ldots + x_n\mathsf{A}_n,$$

with $x = (x_1, \ldots, x_n)$.

Suppose that $\mathsf{A_i} \in \mathbb{Q}^{\mathbf{m}\times\mathbf{m}}$.

Examples

$$\begin{bmatrix} x_{11} & \ldots & x_{1m} \\ \vdots & \ddots & \vdots \\ x_{m1} & \ldots & x_{mm} \end{bmatrix} \quad \begin{bmatrix} x_1 & 1 - x_3 \\ x_2 & 1 \end{bmatrix}$$

Lyapounov Stability is an LMI

$$\boxed{\frac{d\mathbf{f}}{dt} = M \cdot \mathbf{f}} \rightarrow \begin{array}{r} \text{Solve in } P \\ P \succ 0 \\ M^T P + PM \prec 0 \end{array}$$

For $\mathsf{A}_i = \mathsf{A}_i^T$, a linear matrix inequality is the positivity condition

$$\mathsf{A}_0 + x_1\mathsf{A}_1 + \ldots + x_n\mathsf{A}_n \succeq 0$$

where $\succeq 0$ is positive semidefinite.

The set $\mathscr{S} = \{x \in \mathbb{R}^n : A(x) \succeq 0\}$ is called a spectrahedron. Properties:

- convex basic semi-algebraic
- exposed faces
- over $Fr(\mathscr{S}) \rightarrow \det(A) = 0$

$$\begin{bmatrix} 1 & x_1 & 0 & x_1 \\ x_1 & 1 & x_2 & 0 \\ 0 & x_2 & 1 & x_3 \\ x_1 & 0 & x_3 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & x_1 & x_2 & x_3 \\ x_1 & 1 & x_1 & x_2 \\ x_2 & x_1 & 1 & x_1 \\ x_3 & x_2 & x_1 & 1 \end{bmatrix}$$

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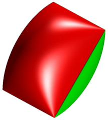
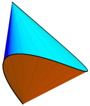




Linear matrices and Spectrahedra

Define the complex determinantal variety

$$\mathcal{D}_r = \left\{ \mathbf{x} \in \mathbb{C}^n \;\middle|\; \text{rank } \mathsf{A}(\mathbf{x}) \leq r \right\} \qquad \text{for } r \leq m-1.$$

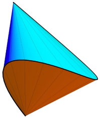
Theorem — Henrion-N.-Safey

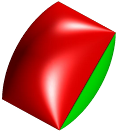
Let $A(\mathbf{x})$ be symmetric. Let $\mathbf{r} = \min \left\{ \text{rank } \mathsf{A}(\mathbf{x}) \mid \mathbf{x} \in \mathscr{S} \right\}$ and C a connected component of $\mathcal{D}_{\mathbf{r}} \cap \mathbb{R}^n$ such that $C \cap \mathscr{S} \neq \emptyset$. Then $C \subset \mathscr{S}$.

Remark: compute points in $\mathcal{D}_{\mathbf{r}} \cap \mathbb{R}^n \to$ points in $\mathscr{S}$.

Semidefinite Programming:

$$\begin{array}{ll} \min & c_1\mathbf{x}_1 + \ldots + c_n\mathbf{x}_n \\ s.t. & \mathbf{x} \in \mathscr{S} = \left\{ x \in \mathbb{R}^n \mid \mathsf{A}(x) \succeq 0 \right\} \end{array}$$





The "probability" to be solution is positive for small-rank points.

Problem statement

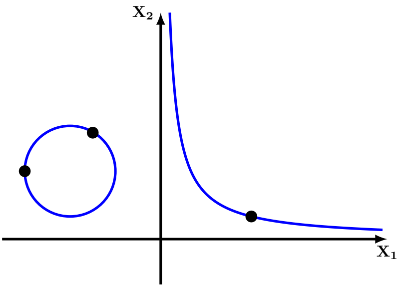
Given

- $m, n, r \in \mathbb{N}$, $0 \leq r \leq m-1$
- $A_0, A_1 \ldots A_n \in \mathbb{Q}^{m \times m}$,

let

- $A(\mathbf{x}) = A_0 + \mathbf{x}_1 A_1 + \ldots + \mathbf{x}_n A_n$
- $\mathcal{D}_r = \left\{ \mathbf{x} \in \mathbb{C}^n \;\middle|\; \text{rank } A(\mathbf{x}) \leq r \right\}$.

$\rightarrow \mathcal{D}_r$ is an algebraic set!



Then

Compute one point in each connected component of $\mathcal{D}_r \cap \mathbb{R}^n$.

* one point in each connected component = a good sample set

* $r = m - 1$: hypersurface $\det A = 0$

* $r = m - 1, n = 1$: Real Eigenvalue Problem

* $n \geq 2$: positive dimensional problem

* first step for solving $\det A > 0$ and $\det A \geq 0$.

State of the art

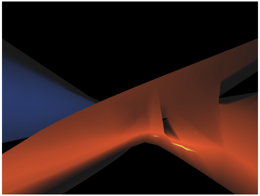
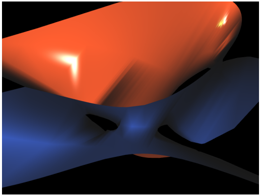
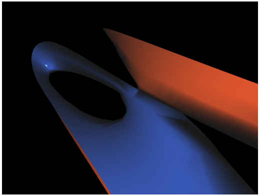
Existence/computation of real roots

- $F(x_1 \dots x_n) = 0$, $\deg F = m$: complexity $\mathbf{m}^{\mathcal{O}(\mathbf{n})}$, hard in practice [Basu, Pollack, Roy, Grigoriev, Vorobjov, Heintz, Solerno];
- Using polar varieties [Bank, Giusti, Heintz, Mbakop, Pardo, Safey, Schost]:
  - $\mathcal{O}(\mathbf{m}^{3\mathbf{n}})$: regular case
  - $\mathcal{O}(\mathbf{m}^{4\mathbf{n}})$: singular case.
- Gröbner Bases = to compute solutions to poly. equations [`FGb`,`RAGlib`]
- Quadratic case. Complexity: poly. in n, expon. in the codimension

Determinantal structure

- Extensively studied in Algebraic Geometry
- Finite (0-dimensional) case: Gröbner Bases methods $\rightsquigarrow$ complexity bounds [Faugère, Safey, Spaenlehauer]

Positive dimensional singular varieties

How to avoid singularities?

Input : $P_1 = \ldots = P_a = 0$
$V(P_1, \ldots, P_a)$ possibly singular

$\longrightarrow$

A system $Q_1 = \ldots = Q_b = 0$
$V(Q_1, \ldots, Q_b)$ good properties

How to reduce the dimension?

$\dim V(Q_1, \ldots, Q_b) > \mathbf{0}$

$\longrightarrow$

A system $R_1 = \ldots = R_c = 0$
$V(R_1, \ldots, R_c)$ is finite

and such that

$$C \subset (V(P_1, \ldots, P_a) \cap \mathbb{R}^n) \longrightarrow C \cap (V(R_1, \ldots, R_c) \cap \mathbb{R}^n) \neq \emptyset$$

Removing singularities

Room – Kempf desingularization : we build the bi-linear system $\mathbf{Q}$

$$A(\mathbf{x}) \cdot \mathbf{Y} = A(\mathbf{x}) \cdot \begin{bmatrix} \mathbf{y}_{1,1} & \cdots & \mathbf{y}_{1,m-r} \\ \vdots & & \vdots \\ \mathbf{y}_{m,1} & \cdots & \mathbf{y}_{m,m-r} \end{bmatrix} = 0.$$

$$U \cdot \mathbf{Y} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

where U has full rank. It defines a set $\mathcal{V}_r = V(\mathbf{Q}) \subset \mathbb{C}^{n+m(m-r)}$.

- rank $A(\mathbf{x}) \leq r \iff \dim(\ker A(\mathbf{x})) \geq m - r$. $\langle \mathbf{Y} \rangle = \ker A(\mathbf{x})$.
- $\Pi_{\mathbf{x}}(\mathcal{V}_r) \subset \mathcal{D}_r$
- Real points in $\mathcal{V}_r$: $(\mathbf{x}, \mathbf{y}) \in \mathcal{V}_r \cap \mathbb{R}^{n+m(m-r)} \to \mathbf{x} \in \mathcal{D}_r \cap \mathbb{R}^n$

Theorem: generic smoothness and equidimensionality of $\mathcal{V}_r = V(\mathbf{Q})$.

$\to$ for generic linear matrices A, the set $\mathcal{V}_r$ has no singular points.

Compute critical points

Consider the projection map

$$\Pi_a(\mathbf{x}, \mathbf{y}) = a_1\mathbf{x}_1 + \ldots + a_n\mathbf{x}_n.$$

Critical points $\to$ solutions to the multi-linear Lagrange system $\mathbf{R}$:

$$\mathbf{Q}(\mathbf{x}, \mathbf{y}) = 0 \qquad \mathbf{z}' \begin{pmatrix} \mathrm{jac}_X \mathbf{Q}(\mathbf{x}, \mathbf{y}) & \mathrm{jac}_Y \mathbf{Q}(\mathbf{x}, \mathbf{y}) \\ a_1, \ldots, a_n & 0 \cdots 0 \end{pmatrix} = 0.$$

where $a = [a_1, \ldots, a_n]^T \in \mathbb{R}^n$.

- $\mathbf{R}$ = Critical points of $\Pi_a(\mathbf{x}, \mathbf{y}) = a_1\mathbf{x}_1 + \ldots + a_n\mathbf{x}_n$ on $\mathcal{V}_r$
- $\mathbf{z}$ = Lagrange multipliers
- $(\mathbf{x}, \mathbf{y})$ critical for $\Pi_a \iff \exists\ \mathbf{z}\ :\ (\mathbf{x}, \mathbf{y}, \mathbf{z})$ is a solution
- # polynomials = # variables

Theorem: generically w.r.t. $\{A_i, a\}$ the solution set is finite.

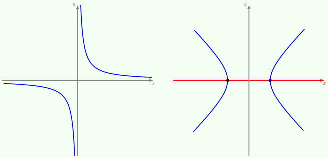
$\to$ for generic projections, finite number of critical points.

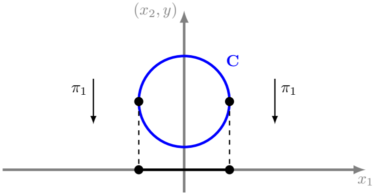
Projections on generic lines

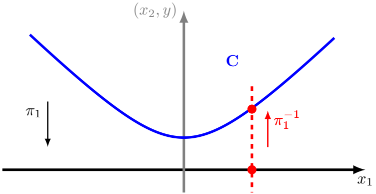
Degenerate situations

Projecting $\{\mathbf{x}\mathbf{y} - 1 = 0\}$ on $\mathbf{y} = 0$ one obtains an open set and no critical points for this map.

change of variables
two critical points







Theorem: generic closedness of projections

$\longrightarrow$ after a change of $\mathbf{x}$−variables, only these two cases can hold.

Output and complexity

The algorithms produces a zero-dimensional ideal $\langle f \rangle = \langle f_1 \dots f_N \rangle$.
Define:

$$\delta := \max_{t=1\dots N} \deg \langle f_1 \dots f_t \rangle$$

Data representation

Rational Parametrization $(p, p_0, p_1, \dots, p_n)$:

$$\mathbf{x}_1 = \frac{p_1(t)}{p_0(t)}, \dots, \mathbf{x}_n = \frac{p_n(t)}{p_0(t)}, \quad p(t) = 0.$$

Complexity model: [Giusti, Lecerf, Salvy, 2001, Geometric Resolution]

the arithmetic complexity is essentially quadratic on δ.

!!! Bézout bounds $\longrightarrow$ δ exponential in m, n

Multi-linear structure $\longrightarrow$ Multi-linear Bézout bounds

Output and complexity

Complexity analysis — Henrion-N.-Safey

The number of arithmetic operations over $\mathbb{Q}$ needed to compute one point per connected component of $\mathcal{D}_r$ with parameters (m, n, r) is in:

- generic linear matrices

$$\mathcal{O}\left(Poly(m,n,r) \cdot \binom{n + m(m-r)}{n}^6 \right)$$

- symmetric linear matrices

$$\mathcal{O}\left(Poly(m,n,r) \cdot \binom{n + \frac{(m+r+1)(m-r)}{2}}{n}^6 \right)$$

- Hankel/Toeplitz linear matrices

$$\mathcal{O}\left(Poly(m,n,r) \cdot \binom{n + 2m - r - 1}{n}^6 \right)$$

An algorithm for spectrahedra

Input $\mathsf{A}_0, \mathsf{A}_1, \ldots, \mathsf{A}_n$ symmetric matrices.

Output

- [ ] if $\mathscr{S} = \emptyset$
- a RP $(p, p_0, p_1, \ldots, p_n) \in \mathbb{Q}[t]$, the min-rank $\mathbf{r}$

Procedure For r from 1 to $m-1$ do

- apply Algorithm to $(\mathsf{A}(\mathbf{x}), r)$;
- for all $\mathbf{x} \in V(p(t), x_i - p_i(t)/p_0(t))$ test whether $\mathbf{x} \in \mathscr{S}$;
- if yes, return $(p, p_0, p_1, \ldots, p_n, r)$.

Timings

Table: $m - r = 1$

m	n	Algorithm	RAGlib
2	4	0.22 s	2.25 s
2	10	0.63 s	25.6 s
2	20	1.99 s	4065 s
3	3	0.49 s	2.8 s
3	20	10.5 s	$\simeq$ 7 h
4	2	0.35 s	0.35 s
4	4	110 s	835 s
4	16	4736 s	∞
4	20	7420 s	∞

Table: $m - r = 2$

m	n	time (s)
3	2	0.23 s
3	8	10.3 s
3	12	175 s
4	4	503 s
4	5	716 s
5	2	3 s
5	3	7 s

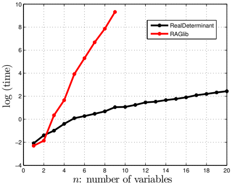


Figure: $(k, r) = (3, 2)$

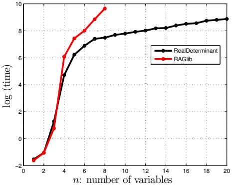


Figure: $(k, r) = (4, 3)$

First implementation under Maple.
We use FGb for Grobner bases computations

Number of complex solutions

$A_0, A_1, \ldots, A_n$ are random or random-symmetric

m	n	r	generic	symm.	m	n	r	generic	symm.
2	2	1	4	4	3	9	2	39	26
2	3	1	6	5	3	15	2	39	26
2	4	1	6	5	3	20	2	39	26
2	8	1	6	5	4	3	3	52	42
2	20	1	6	5	4	4	3	120	80
3	3	2	21	17	4	6	3	264	152
3	4	2	33	23	4	7	3	284	162
3	5	2	39	26	4	10	3	284	162
3	6	2	39	26	4	20	3	284	162

$$\textit{generic} \leq \sum_{N=(m-r)^2}^{\min\{n, m^2-r^2\}} \sum_{\ell=0}^{r(m-r)} \binom{m(m-r)}{N-\ell} \binom{N-1}{N-(m-r)^2-\ell} \binom{r(m-r)}{\ell}$$

$$\textit{symmetric} \leq \sum_{N=c-r(m-r)}^{\min\{n, c+r(m-r)\}} \sum_{\ell=0}^{r(m-r)} \binom{c}{n-\ell} \binom{n-1}{n-c+r(m-r)-\ell} \binom{r(m-r)}{\ell}$$

$$\text{with } c = \frac{(m-r)(m+r+1)}{2}$$

Thank you