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HARMONICS SUMS AND POLYLOGARITHMS AT NON-POSITIVE INTEGERS

Marseille - 2014

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- Harmonic sums $H^{-}_{y_{s_1}\ldots y_{s_r}}(N) = \sum_{N \geq n_1 > \ldots > n_r > 0} n_1^{s_1} \ldots n_r^{s_r}, \quad N > 0.$

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- The values of ζ at negative integers, by analytic prolongation.

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Symbols

- Y : the infinite alphabet $\{y_i\}_{i\geq 1}$ equipped with the order $y_1 > y_2 > \ldots,$
- Y^* : the set of words over Y .
- One constructs a map $1-1$ between Y^* and $\sqcup_{n\in\mathbb{N}}(\mathbb{N}_+)^n$, namely $w = y_{s_1}\ldots y_{s_r} \leftrightarrow (s_1,\ldots,s_r)$.
- $X = \{x_0 < x_1\}$: the finite alphabet,
- X^* : the set of words over X.
- X^* and $\sqcup_{n\in\mathbb{N}}(\mathbb{N})^n$ be a morphism $1-1$ defines by $x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1 \mapsto (s_1,\ldots,s_r)$.

1. Harmonic sums

- Let $r \in \mathbb{N}_+$, and for all $s_1, \ldots, s_r \in \mathbb{N}_+$, one defines the harmonic sums by the formula

$$H^{-}_{y_{s_1} y_{s_2} \ldots y_{s_r}}(N) = \sum_{N \geq n_1 > \ldots > n_r > 0} n_1^{s_1} \ldots n_r^{s_r}, \forall N \in \mathbb{N}_+. \quad (2)$$

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- The Bernoulli numbers are a sequence of real numbers, defined by the following formula

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}. \quad (3)$$

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$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}. \quad (3)$$

- When $r = 1$, for $s_1 \geq 1$, one has

$$H^{-}_{y_{s_1}}(N) = \frac{1}{s_1 + 1} \sum_{n_1=0}^{s_1} \binom{s_1 + 1}{n_1} B_{n_1} (N+1)^{s_1 + 1 - n_1}. \quad (4)$$

for any $N \in \mathbb{N}_+$.

1. Harmonic sums

Proposition.

Let $w = y_{s_1} y_{s_2} \in Y^*$, then

$$H_w^-(N) = \sum_{k_1=0}^{s_2} \sum_{k_2=0}^{p-1-k_1} \sum_{k_3=0}^{p-k_1-k_2} \frac{B_{k_1} B_{k_2}}{(s_2+1)(p-k_1)} \binom{s_2+1}{k_1} \binom{p-k_1}{k_2} \binom{p-k_1-k_2}{k_3} N^{k_3}.$$

where $p = (w) + |w|$ and $N \in \mathbb{N}^*$.

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where $p = (w) + |w|$ and $N \in \mathbb{N}^*$.

- For example,

$$H_{y_2 y_2}^-(N) = \frac{1}{360} N(N-1)(2N+1)(2N-1)(5N+6)(N+1),$$

$$H_{y_2 y_3}^-(N) = \frac{1}{840} N(N-1)(N+1)(30N^4 + 35N^3 - 33N^2 - 35N + 2),$$

$$H_{y_2 y_4}^-(N) = \frac{N(N-1)(N+1)(63N^5 + 72N^4 - 133N^3 - 138N^2 + 49N}{2520}$$

$\dots \quad \dots \quad \dots$

Theorem

For any $w = y_{s_1} y_{s_2} \dots y_{s_r} \in Y^*$, then $H_w^-(N)$ is a polynomial of degree $(w) + |w|$ of N.

1.Harmonic sums

Theorem

For any $w = y_{s_1} y_{s_2} \ldots y_{s_r} \in Y^*$, then $H_w^-(N)$ is a polynomial of degree $(w) + |w|$ of N.

- Hence, there is a non-zero constant, namely C_w^-, which only depends to w and r such that, when $N \to \infty$

$$H_w^-(N) \sim N^{(w)+|w|} C_w^-, \tag{5}$$

i.e,

$$\lim_{N\to\infty} \frac{H_w^-(N)}{N^{(w)+|w|} C_w^-} = 1. \tag{6}$$

1.Harmonic sums

Proposition

$$C_w^- = \prod_{w=uv;v\neq 1_{Y^*}} \frac{1}{(v) + |v|}. \qquad (7)$$

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$$C_w^- = \prod_{w=uv;v\neq 1_{Y^*}} \frac{1}{(v)+|v|}. \quad (7)$$

-

w	C_w^-
y_4	$\frac{1}{5}$
y_{10}	$\frac{1}{11}$
y_2y_1	$\frac{1}{8}$
y_2^2	$\frac{1}{15}$
y_2y_{10}	$\frac{1}{143}$
$y_2^2y_3$	$\frac{1}{280}$
y_2^4	$\frac{1}{1944}$
$y_2^4y_{20}$	$\frac{1}{13471920}$
$y_2^2y_3y_2^2$	$\frac{1}{137440}$
....	...

Proposition

For all $w_1 = y_{s_1} \ldots y_{s_u}; w_2 = y_{t_1} \ldots y_{t_v} \in Y^*$ then

$$H^-_{w_1}(N) H^-_{w_2}(N) = H^-_{w_1 \boxplus w_2}(N). \tag{8}$$

1.Harmonic sums

Proposition

For all $w_1 = y_{s_1} \ldots y_{s_u}; w_2 = y_{t_1} \ldots y_{t_v} \in Y^*$ then

$$H^-_{w_1}(N) H^-_{w_2}(N) = H^-_{w_1 \uplus w_2}(N). \tag{8}$$

- For example, we have that

$$\begin{aligned} H^-_{y_n}(N) H^-_{y_m}(N) &= \sum_{k=1}^{N} k^n \sum_{l=1}^{N} l^m, &(9)\\ &= \sum_{k=1}^{N} \sum_{l=1}^{k-1} k^n l^m + \sum_{k=1}^{N} \sum_{l=k+1}^{N} k^n l^m + \sum_{k=1}^{N} k^{m+n}, &(10)\\ &= H^-_{y_n y_m}(N) + H^-_{y_m y_n}(N) + H^-_{y_{m+n}}(N). &(11) \end{aligned}$$

Note that $w = y_{s_1} \ldots y_{s_r} \mapsto C_w^-$ can not be extended linearly on $\mathbb{Q}\langle Y \rangle$. To see this, we take the example

$$\frac{1}{9} = C^-_{y_2 ⧢ y_2} = C^-_{y_4 + 2y_2^2} \neq C^-_{y_4} + C^-_{2y_2^2} = \frac{1}{5} + \frac{1}{9}. \quad (12)$$

Note that $w = y_{s_1} \ldots y_{s_r} \mapsto C^-_w$ can not be extended linearly on $\mathbb{Q}\langle Y\rangle$. To see this, we take the example

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- From $w = y_{s_1} \ldots y_{s_r} \mapsto H^-_w(N)$ is linear. And so, if we put
$$\mathcal{H}_p = \mathbb{Q}_+\langle\{w \in Y^* | (w) + |w| = p\}\rangle = \mathbb{Q}_+\langle G_p\rangle,$$
then the function C^- such that $C^-((w)) = C^-_w$ is linear on $\mathcal{H}_p$, for any $p \in \mathbb{N}$.

Proposition

Let $d_p = cardG_p$ for all $p \leq 0$. Then we have

$$d_p = \begin{cases} 1 & \Leftrightarrow p = 0 \\ 0 & \Leftrightarrow p = 1 \\ F_{p-2} & \Leftrightarrow p \geq 2 \end{cases} \quad (13)$$

where F_{p-2} is the $(p-2)-th$ Fibonacci's number.

Example.

p	G_p	$cardG_p$	p	G_p	$cardG_p$
0	1_{Y^*}	1	4	y_3, y_1^2	2
1	$\varnothing$	0	5	y_4, y_2y_1, y_1y_2	3
2	y_1	1	6	$y_5, y_3y_1, y_2^2, y_1y_3, y_1^3$	5
3	y_2	1	7	$y_6, y_4y_1, y_3y_2, y_2y_3, y_2y_1^2, y_1y_4, y_1y_2y_1, y_1^2y_2$	8

1.Harmonic sums

Lemma

Let $\alpha : Y^* \to \mathbb{N}$ such that $\begin{cases} \alpha(uv) & = \alpha(u) + \alpha(v) \\ \alpha(1) & = 0 \end{cases}$ then

$$\sum_{w \in Y^*} t^{\alpha(w)} w = \frac{1}{1 - \sum_{y \in Y} t^{\alpha(y)} y}. \qquad (14)$$

- **Proof of proposition.** We see that $l_* : Y^* \to \mathbb{N}$ such that $l_*(w) = (w) + |w|$, then l_* satisfies the above lemma. And so we have

$$\sum_{p \geq 0} \sum_{w \in G_p} t^p w = \sum_{w \in Y^*} t^{l_*(w)} w = \frac{1}{1 - \sum_{s \geq 1} t^{l_*(y_s)} y_s} = \frac{1}{1 - \sum_{s \geq 1} t^{s+1} y_s}$$

$$\xrightarrow{y_s \to 1_{Y^*}} \frac{1}{1 - \sum_{s \geq 1} t^{s+1}} = \frac{1}{1 - \frac{t^2}{1-t}} = \frac{1-t}{1-t-t^2} = 1 + \sum_{n \geq 0} F_n t^{n+2}.$$

- Let $P = \sum_{w \in Y^*} \alpha_w w$, with $\alpha_w \in \mathbb{Q}$, One classically defines its support as $supp(P) = \{w \in Y^* | \alpha_w \neq 0\}$.

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- For $P \in \mathbb{Q}_+ \langle Y \rangle$, i.e, $P = \sum_{w \in Y^*} \alpha_w w, \alpha_w \in \mathbb{Q}_+$,

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$$C_P^- = \sum_{c \in J} \alpha_c C_c^- \tag{15}$$

where $J = \{v \in supp(P) | (v) + |v| = \max\{(u) + |u| \, |u \in supp(P)\}\}$.

Proposition

Let any $T \in \mathbb{Q}_+ \langle Y \rangle$, let $m_T = \max\{(u) + |u| \, |u \in supp(T)\}$. Then

$$H_P^-(N) \sim C_P^- N^{m_P} \quad , \quad H_Q^-(N) \sim C_Q^- N^{m_Q}$$
$$H_{P \uplus Q}^-(N) \sim C_P^- C_Q^- N^{m_{P \uplus Q}}$$
$$C_{P \uplus Q}^- = C_P^- C_Q^-;$$

for any $P, Q \in \mathbb{Q}_+ \langle Y \rangle$.

- One defines the polylogarithm for $r \geq 1, s_1, \ldots, s_r \in \mathbb{N}^*$ and $|z| < 1$ by the following formula

$$Li^{-}_{x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1}(z) = Li^{-}_{s_1,\ldots,s_r}(z) = \sum_{n_1>n_2>\ldots>n_r>0} n_1^{s_1} \ldots n_r^{s_r} z^{n_1}. \qquad (16)$$

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- One notes that

$$\mathrm{Li}^{-}_{x_0^r}(z) = \sum_{n_1>\ldots>n_r>0} z^{n_1} = \left(\frac{z}{1-z}\right)^r. \tag{17}$$

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- One notes that

$$Li^-_{x_0^r}(z) = \sum_{n_1>\ldots>n_r>0} z^{n_1} = \left(\frac{z}{1-z}\right)^r. \quad (17)$$

For example, when $r = 2$, we have $Li^-_{x_0^2}(z) = \sum_{n_1>n_2>0} z^{n_1} = \dfrac{z^2}{(1-z)^2}$.

Proposition

$$\frac{\mathrm{Li}^-_{x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1}(z)}{1-z} = \sum_{N\geq 0} H^-_{y_{s_1}\ldots y_{s_r}}(N) z^N. \tag{18}$$

2. Polylogarithms

Proposition

$$\frac{Li^-_{x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1}(z)}{1-z} = \sum_{N\geq 0} H^-_{y_{s_1}\ldots y_{s_r}}(N) z^N. \quad (18)$$

- For example, one has

$$\frac{1}{1-z} Li^-_{x_0^{n-1}x_1}(z) = \frac{1}{1-z}\sum_{k=1}^{\infty} k^n z^k, \quad (19)$$

$$= \sum_{l=0}^{\infty} z^l \sum_{k=1}^{\infty} k^n z^k, \quad (20)$$

$$= \sum_{k=1}^{\infty} k^n z^k + \sum_{k=1}^{\infty} k^n z^{k+1} + \ldots + \sum_{k=1}^{\infty} k^n z^{k+m} + \ldots, \quad (21)$$

$$= z + (1^n + 2^n) z^2 + (1^n + 2^n + 3^n) z^3 + \ldots., \quad (22)$$

$$= \sum_{N>0} H^-_{y_n}(N) z^N. \quad (23)$$

2. Polylogarithms

- Let $n \in \mathbb{N}^*$, one defines the Euler polynomials by

$$A_n(z) = \sum_{k=0}^{n-1} A_{n,k} z^k, \tag{24}$$

where $A_{n,k}$ is the Eulerian number which is defined by

$$A_{n,k} = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k+1-j)^n. \tag{25}$$

Note that $A_0(z) = 1$.

- $$\mathrm{Li}^-_{x_0^{n-1}x_1}(z) = \frac{zA_n(z)}{(1-z)^{n+1}}, \tag{26}$$

for all n.

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$$Li^-_{x_0^{n-1}x_1}(z) = \frac{zA_n(z)}{(1-z)^{n+1}}, \tag{26}$$

for all n.

- Hence, for any $n \in \mathbb{N}_+$, $Li^-_{x_0^{n-1}x_1}(z)$ is a polynomial of $\dfrac{1}{1-z}$ of degree $n+1$ over $\mathbb{Q}$.

2. Polylogarithms

- $$Li^{-}_{x_1}(z) = -\frac{1}{1-z} + \frac{1}{(1-z)^2}. \quad (27)$$
  $$Li^{-}_{x_0x_1}(z) = \frac{1}{1-z} - \frac{3}{(1-z)^2} + \frac{2}{(1-z)^3}. \quad (28)$$
  $$Li^{-}_{x_0^2x_1}(z) = -\frac{1}{1-z} + \frac{7}{(1-z)^2} - \frac{12}{(1-z)^3} + \frac{20}{(1-z)^4}. \quad (29)$$

- $$\begin{aligned} Li^{-}_{x_1}(z) &= -\frac{1}{1-z} + \frac{1}{(1-z)^2}. && (27)\\ Li^{-}_{x_0x_1}(z) &= \frac{1}{1-z} - \frac{3}{(1-z)^2} + \frac{2}{(1-z)^3}. && (28)\\ Li^{-}_{x_0^2x_1}(z) &= -\frac{1}{1-z} + \frac{7}{(1-z)^2} - \frac{12}{(1-z)^3} + \frac{20}{(1-z)^4}. && (29) \end{aligned}$$

- Moreover,

$$Li^{-}_{x_0^{n-1}x_1}(z) = \sum_{k=0}^{n+1}\sum_{j=0}^{n-1} A_{n,j}\binom{j+1}{k}(-1)^k \frac{1}{(1-z)^{n+1-k}}. \qquad (30)$$

Proposition

Let $\iota_1 = \int \frac{*d}{(1-z)dz}; \theta_0 = z\frac{d}{dz}$ be operators . For any $s_1, .., s_r$ and $r \in \mathbb{N}$,

$$Li^{-}_{x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1}(z) = \left(\theta_0^{s_1+1}\iota_1\right)\ldots\left(\theta_0^{s_{r-1}+1}\iota_1\right) Li^{-}_{x_0^{s_r-1}x_1}(z). \quad (31)$$

Theorem

For any $w = x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1 \in X^*$ and $r \in \mathbb{N}$, $Li^{-}_{w}(z)$ is a polynomials of $\frac{1}{1-z}$ of degree $(w) + |w|$.

For example,

$$\mathrm{Li}^{-}_{x_0^{m-1}x_1x_0^{n-1}x_1}(z) \quad = \quad (\theta_0^{m+1}\iota_1)(\mathrm{Li}_{-n}(z)),$$

and so

$$= \sum_{k=0}^{n-1}\sum_{j=0}^{k+1} A_{n-1,j}\binom{k+1}{j}(-1)^j\left[\theta_0^m\frac{1}{(1-z)^{n+1-j}} - \theta_0^m\frac{1}{(1-z)^{n-j}}\right]$$

where,

$$\theta_0^k\left(\frac{1}{(1-z)^s}\right) = \sum_{j=0}^{k}(-1)^{k-j}\frac{\sum_{u\in I_j^k\cup I_{j+1}^k} s(u)}{(1-z)^{s+j}} \tag{32}$$

with $I_j^k = \left\{(i_1,\ldots,i_j)\middle| i_t > 0, \forall\, t = 1,\ldots,j; \sum i_t = k\right\}$, and $s(i_1,..,i_j) = s^{i_1}(s+1)^{i_2}\ldots(s+j-1)^{i_j}$, for any $s\in\mathbb{N}_+$.

Hence, there is a constant B_w^- such that

$$\lim_{z\to 1^-} \frac{(1-z)^{(w)+|w|} Li_w^-(z)}{B_w^-} = 1. \qquad (33)$$

Proposition

For any $w = x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1 \in X^*$ and $r \geq 2$, then

$$B_w^- = A_{s_r}(1) \prod_{k=2}^{r} \left[\prod_{i=k}^{s_{r-k+1}+k-1} (s_r + \ldots + s_{r-k+2} + i) \right]. \qquad (34)$$

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- For example, for $r = 1$, one has $B^-_{x_0^{s_1-1}x_1} = A_{s_1}(1)$.

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- For example, for $r = 1$, one has $B^-_{x_0^{s_1-1}x_1} = A_{s_1}(1)$.
- And for $r = 2$, one has

$$B^-_{x_0^{s_1-1}x_1x_0^{s_2-1}x_1} = \left[A_{(s_2)}(1)\right](s_1+2)(s_1+3)\ldots(s_1+s_2+1), \tag{35}$$

for all $s_1, s_2 \in \mathbb{N}_+$.

3. Morphism algebras

Proposition

The map H^- from $(\mathbb{C}\langle Y\rangle, ⧢)$ to $\left(\mathbb{C}\{H_w^-\}_{w\in Y_*}, .\right)$ such that $H^-(w) = H_w^-$ for $w \in Y^*$ is a morphism of algebras.

Now, for any $w \in Y^*$, let $P_w^-(z) := \dfrac{Li_w^-(z)}{1-z} = \sum_{N\geq 0} H_w^-(N) z^N$, then

Proposition

The map P^- from $(\mathbb{C}\langle Y\rangle, ⧢)$ to $\left(\mathbb{C}\{P_w^-\}_{w\in Y_*}, \odot\right)$ such that $P^-(w) = P_w^-$ for any $w \in Y^*$ is a morphism of algebras.

4. An example of values of zeta function at non-positive integers

- Let $s_1, s_2, s_3, s_4 \in \mathbb{Z}_-$, one denotes that

$$\zeta(s_1, \ldots, s_4) = \sum_{0<k_1<k_2<k_3<k_4} \frac{1}{k_1^{s_1} \ldots . k_4^{s_4}}. \tag{36}$$

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- We can write again to the forme

$$\zeta(s_1, \ldots, s_4) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{m_3=0}^{\infty} \sum_{m_4=0}^{\infty} \frac{1}{(1+m_1)^{s_1} \ldots . (4+m_1+\ldots+m_4)^{s_4}}. \tag{37}$$

4. An example of values of zeta function at non-positive integers

- Put that $a = (1, 2, 3, 4)$, $b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$, $s = (s_1, s_2, s_3, s_4)$ and

$$B = \{(1,1),(2,1),(2,2),(3,1),(3,2),(3,3),(4,1),(4,2),(4,3),(4,4)\}. \tag{38}$$

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- Now, let G_4 be the symmetric group of 4 elements. For each of $w \in G_4$, we set

$$B_w = \{(w(n), r) | (n, r) \in B\} \tag{39}$$

and

$$\nu_{w,r} = \max\{n | (n, r) \in B_w\}; L_w = \{\nu_{w,r} | r \in \{1, 2, 3, 4\}\}. \tag{40}$$

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- On the set $\{1, 2, 3, 4\}$, Let $>_w$ be an order relation for any $w \in G_4$, one defines $>_w^*$ by

$$m >_w^* n \quad \Leftrightarrow \quad w(m) >_w w(n). \tag{41}$$

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Then $\sim$ is an equivalent relation on G_4.

4. An example of values of zeta function at non-positive integers

- For any $w \in [G_4] = G/\sim$ and $r, n, m \in \mathbb{N}$,

$$\eta_{w,r} = w^{-1}(\nu_{w,r}), \tag{43}$$

$$\kappa_{w,n} = \left|\{r | 1 \le r \le 4 : n >_w^* \eta_{w,r}\}\right|. \tag{44}$$

$$y_{w,m} = \prod_{n >_w^* m} \alpha_n, \forall \alpha = (\alpha_1, \dots, \alpha_4) \in \mathbb{C}^4. \tag{45}$$

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- Defines

$$G_w(t_w; \alpha) = \frac{\alpha_1^{t_{w,1}-1} ... \alpha_4^{t_{w,4}-1}}{(e^{y_{w,1}+...y_{w,4}} - 1)...(e^{y_{w,4}} - 1)}. \quad (46)$$

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- One defines the multiple Bernoulli numbers by

$$G_w(t_w; \alpha) = \sum_{k_1=0}^{\infty} \ldots \sum_{k_4=0}^{\infty} \frac{B_w(k)}{k_1! \ldots k_4!} \alpha_1^{k_1} \ldots \alpha_4^{k_4}, \forall k = (k_1, \ldots, k_4). \tag{47}$$

4. An example of values of zeta function at non-positive integers

- By Y.Komori, one has

$$\zeta(\overbrace{-u}^{w}) = (-1)^{u_1+\ldots+u_4} u_1!\ldots u_4! \frac{B_w(k)}{k_1!\ldots k_4!} \qquad (48)$$

where $k_n = \kappa_{w,n} + \sum_{n >_w^* m} u_m$ for any $u \in \mathbb{N}^4$ and $w \in [G_4]$.

- Moreover,

$$\zeta(\underbrace{-u}_{x}) = \sum_{w \in [G_4]} \zeta(\overbrace{-u}^{w}) \prod_{n=1}^{4} \frac{x_n}{\sum_{n >_w^* m} x_m}, \qquad (49)$$

where $x \in \mathbb{N}^4$.

- Let $>_w$ be the weakest order relation (in Y. Komori), we has

$$\begin{aligned}[G_4] &= \{id; (34), (234), (24), (243), (1234), (124), (1243), (134), \}\\ &\cup \{(1324), (13)(24), (143), (1432), (142), (1423), (14)(23)\}\end{aligned}$$

4. An example of values of zeta function at non-positive integers

- Take $>_w$ is the weakest order relation then we have that $id \in [G_4]$.
- If $w = id$ then $B_{id} = B = \{(1,1),(2,1),(2,2),(3,1),(3,2),(3,3),(4,1),(4,2),(4,3),(4,4)\}$, and so $\nu_{id} = (4,4,4,4)$; $\eta_{id} = (4,4,4,4)$; $\kappa_{id} = (0,0,0,4)$; $y_{id} = (\alpha_1\alpha_4, \alpha_2\alpha_4, \alpha_3\alpha_4, \alpha_4)$.
- And so

$$G_{id}(\alpha) = \frac{\alpha_4^4}{(e^{\alpha_4(1+\alpha_1+\alpha_2+\alpha_3)} - 1)(e^{\alpha_4(1+\alpha_2+\alpha_3)} - 1)(e^{\alpha_4(1+\alpha_3)} - 1)(e^{\alpha_4} - 1)} \tag{50}$$

- Hence, $B_{id}(0,0,0,4) = \frac{251}{30}$, and so $\zeta(\overbrace{0,0,0,0}^{id}) = \frac{251}{720}$.

Conclusions

- We proved that $H^-_{y_{s_1}\dots y_{s_r}}(N)$ is a polynomial of degree $s_1 + \dots + s_r + r$ of N and we give the explicit formula to compute the constants $C^-_{y_{s_1},\dots,y_{s_r}}$ such that

$$\lim_{N\to\infty} \frac{C^-_{y_{s_1},\dots,y_{s_r}} N^{s_1+\dots+s_r+r}}{H^-_{y_{s_1},\dots,y_{s_r}}(N)} = 1. \tag{51}$$

- We prove also that $Li^-_{x_0^{s^1-1}x_1\dots,x_0^{s^r-1}x_1}(z)$ is a polynomial of degree $s_1 + \dots + s_r + r$ of $(1-z)^{-1}$ and we give the explicit formula to compute the constants $B^-_{x_0^{s_1-1}x_1\dots x_0^{s_r-1}x_1}$ such that

$$\lim_{z\to 1^-} \frac{B^-_{x_0^{s_1-1}x_1\dots x_0^{s_r-1}x_1}(1-z)^{-(s_1+\dots+s_r+r)}}{Li^-_{x_0^{s^1-1}x_1\dots,x_0^{s^r-1}x_1}(z)} = 1. \tag{52}$$

- We give the shuffle structure for harmonic sums.
- We compute an example about the values of multiple zeta at negative integers, by analytic prolongation.

Thank you very much!