Précision p-adique

Journées Nationales du Calcul Formel 2014

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Some examples of essentially p-adic algorithms

- Polynomial factorization with Hensel lemma ;
- Kedlaya's counting-point algorithm on hyperelliptic curves with p-adic cohomology ;

1. p-adic precision
   - p-adic algorithms and step-by-step analysis
   - The differential approach
   - Improvements

2. Precision in practice
   - Linear Differential Equation (cf BGPS)
   - About implementation
   - Lifting method

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Definition of the precision

Finite-precision p-adics

Elements of $\mathbb{Q}_p$ can be written $\sum_{i=-l}^{+\infty} a_i p^i$, with $a_i \in [\![0, p-1]\!]$, $l \in \mathbb{Z}$ and p a prime number.
While working with a computer, we usually only can consider the beginning of this power serie expansion: we only consider elements of the following form $\boxed{\sum_{i=l}^{d-1} a_i p^i + O(p^d)}$, with $l \in \mathbb{Z}$.

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Example

The order of $3 * 7^{-1} + 4 * 7^0 + 5 * 7^1 + 6 * 7^2 + O(7^3)$ is 3.

p-adic precion *vs* real precision

The quintessential idea of the step-by-step analysis is the following :

Proposition (p-adic errors don't add)

Indeed,

$$(a + O(p^k)) + (b + O(p^k)) = a + b + O(p^k).$$

That is to say, if a and b are known up to precision $O(p^k)$, then so is $a + b$.

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Remark

It is quite the opposite to when dealing with real numbers, because of **Round-off error** :

$$(1 + 5 * 10^{-2}) + (2 + 6 * 10^{-2}) = 3 + 1 * 10^{-1} + 1 * 10^{-2}.$$

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Proposition (addition)

$$(x_0 + O(p^{k_0})) + (x_1 + O(p^{k_1})) = x_0 + x_1 + O(p^{min(k_0,k_1)})$$

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Proposition (division)

$$\frac{xp^a + O(p^b)}{yp^c + O(p^d)} = x * y^{-1} p^{a-c} + O(p^{\min(d+a-2c,b-c)})$$

In particular,

$$\frac{1}{p^c y + O(p^d)} = y^{-1} p^{-c} + O(p^{d-2c})$$

Optimality

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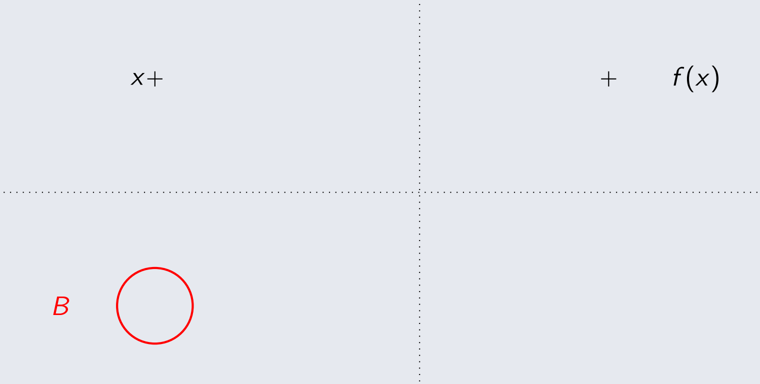
Geometrical meaning

Interpretation



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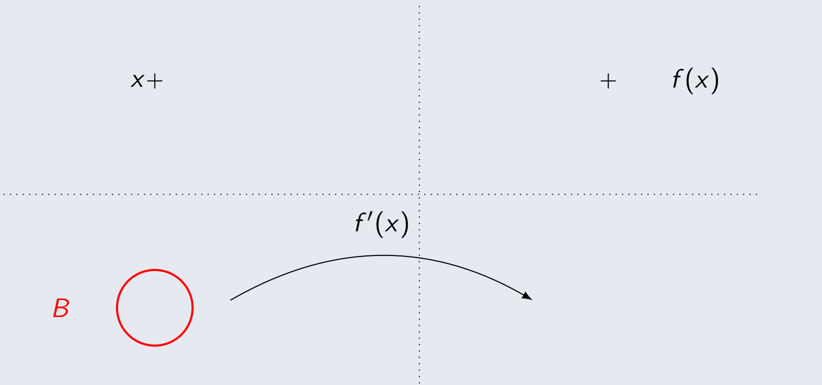

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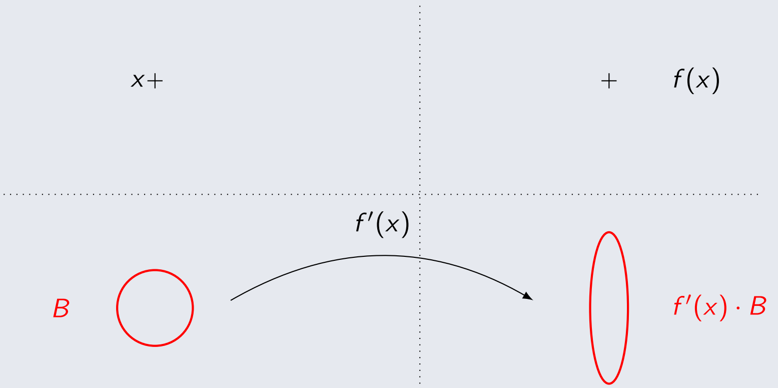

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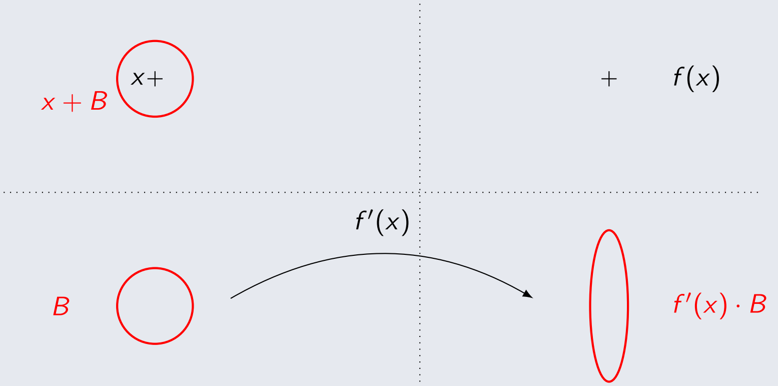

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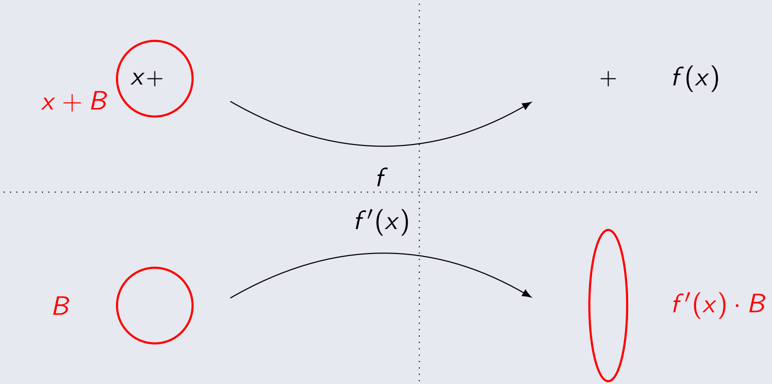

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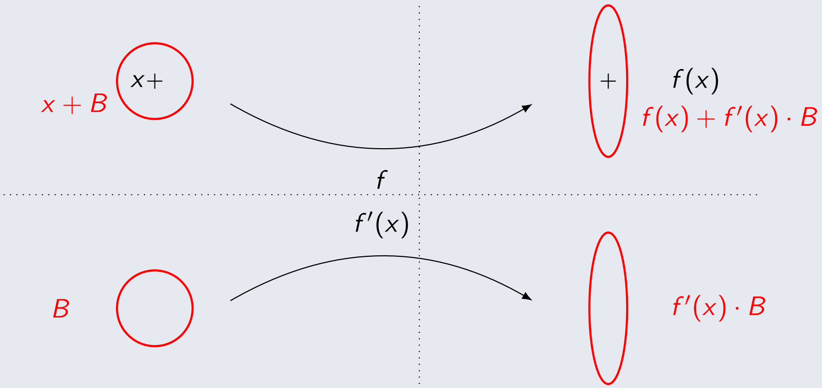

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Remark

Our framework can be extended to **(complete) ultrametric K-vector spaces** (*e.g.* $\mathbb{F}_p((X))^n$, $\mathbb{Q}((X))^m$).

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This can be determined with **Newton-polygon** techniques.

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Newton series

Definition

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Remark

When $char(k) = 0$, by Newton's identities, we can recover P from f_P.

Computation over Newton series

Proposition

Let $P \times Q = \prod_i (X - \alpha_i) \prod_j (X - \beta_j)$ *and* $P \otimes Q = \prod_{i,j} (X - \alpha_i \beta_j)$.

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Remark

How to do that on $\mathbb{Z}/p\mathbb{Z}$? We can lift $\mathbb{Z}/p\mathbb{Z}$ to $\mathbb{Z}_p$, and reduce mod p afterwards.

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with $a(x) = -f_P$ *and* $y = P^*$. *Thus, we can recover* P *with* f_P *by solving a linear differential equation (in* $\mathbb{Z}_p[[x]]$*).*

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Since $\int \sum_i a_i x^i = \sum_i a_i \frac{x^{i+1}}{i+1}$, it yields a logarithmic loss in precision.

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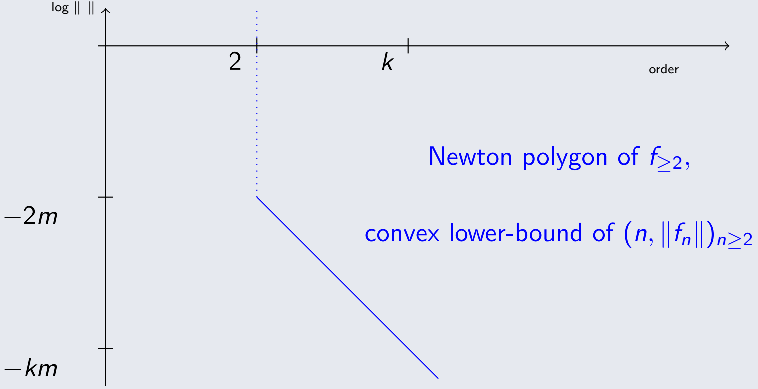

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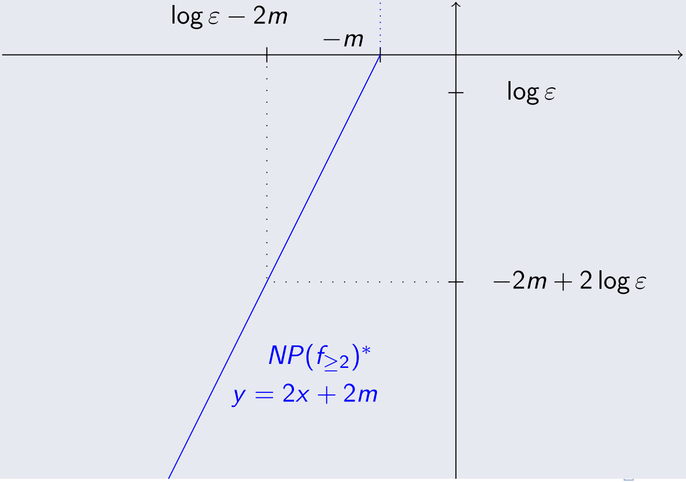

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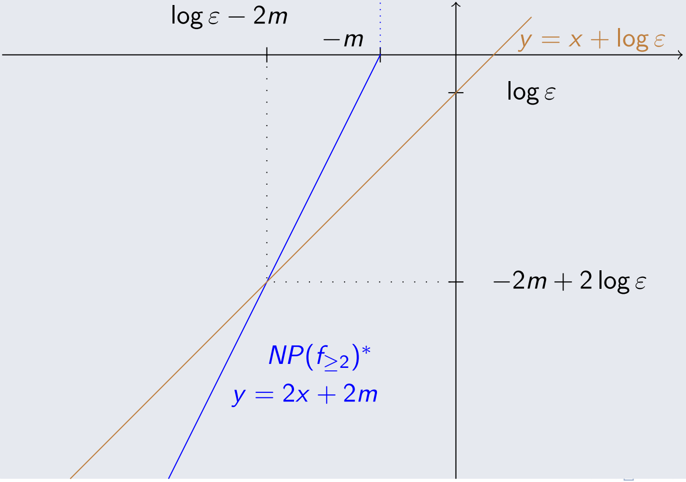

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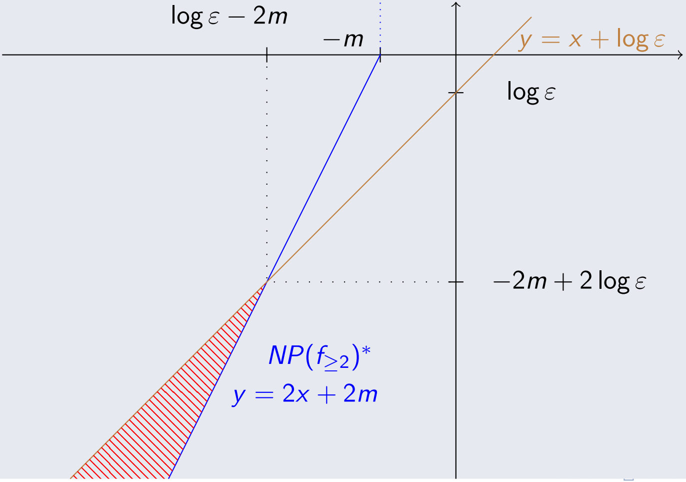

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Newton scheme

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One can compute y_0 through some Newton scheme :

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One lose $E(\log_p(2^l))$ at each step. There are $E(\log_2(2^N))$ steps to reach x^{2^N}, it means a **naïve** loss in precision to compute the coefficient of x^{2^N} in $\left((\log 2^N)^2\right)$.

What happens in practice ?

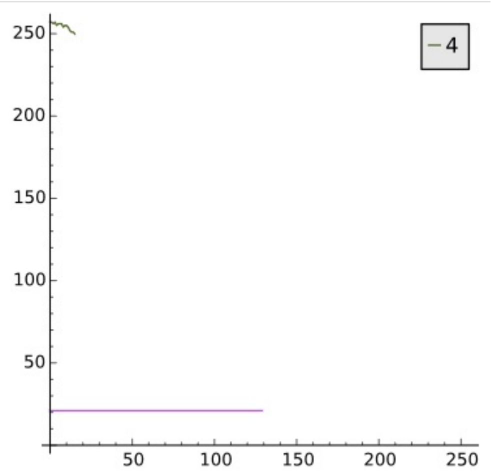
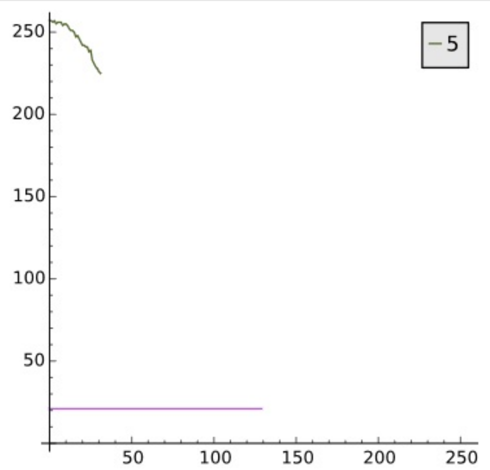


Figure: Precision over the output

What happens in practice ?



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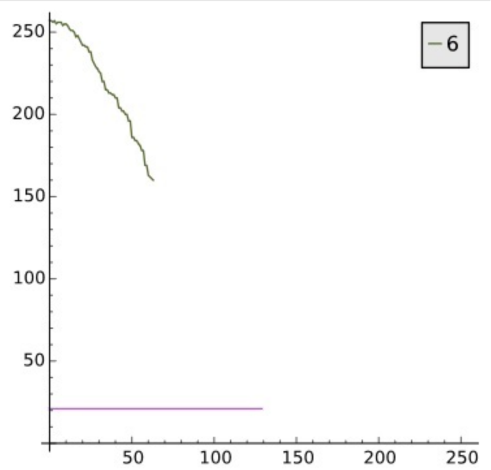

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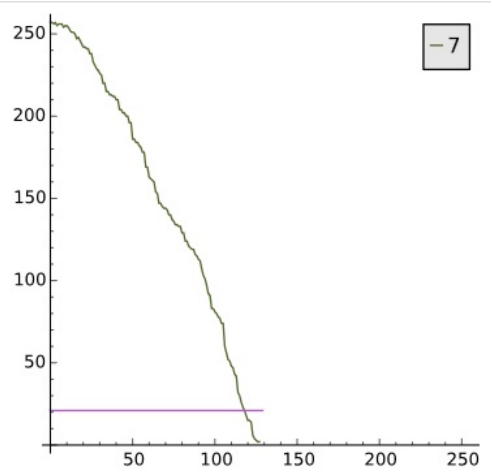

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$x + O(p^N)$

$x+$

B

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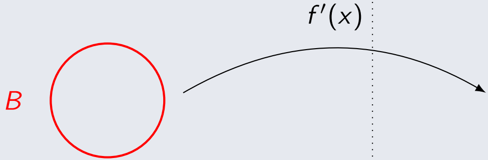

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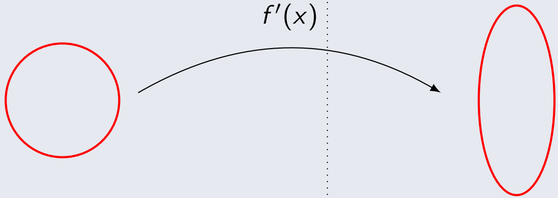

$f'(x) \cdot B$

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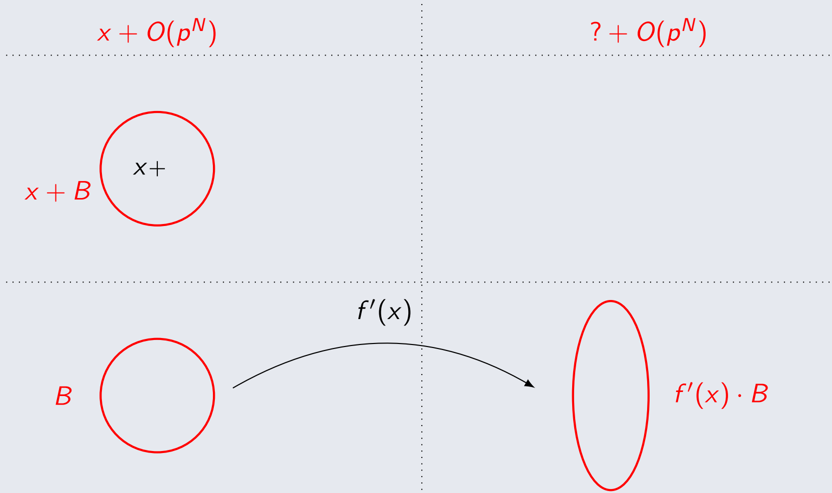

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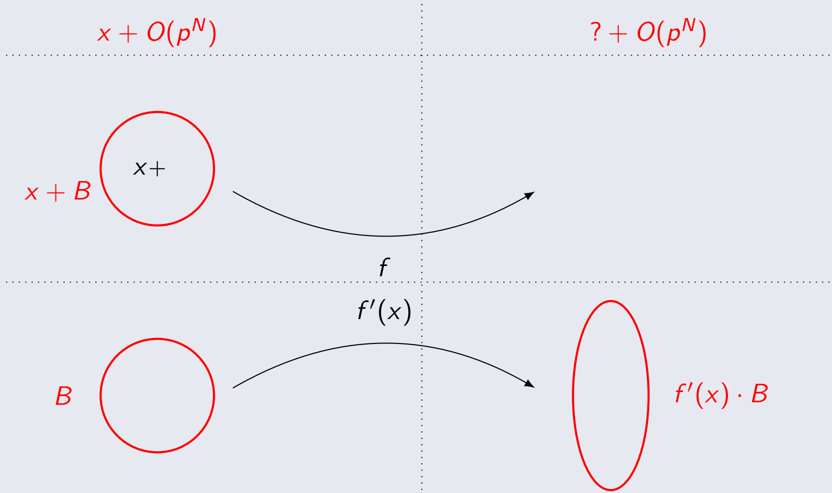

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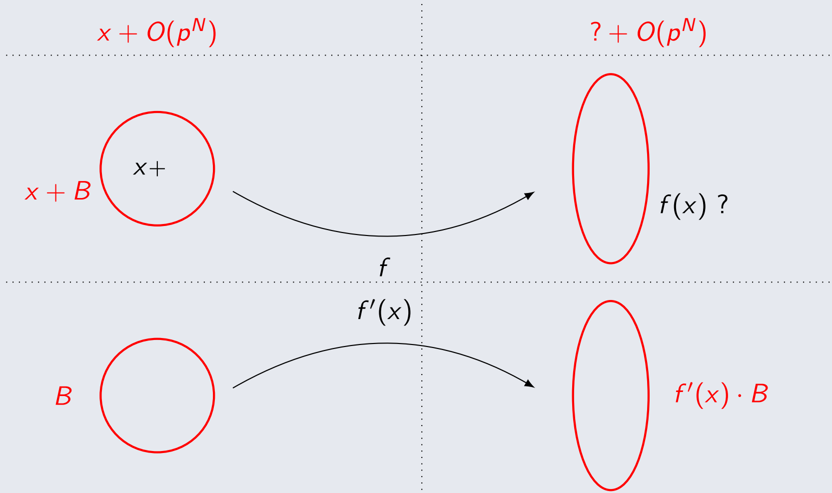

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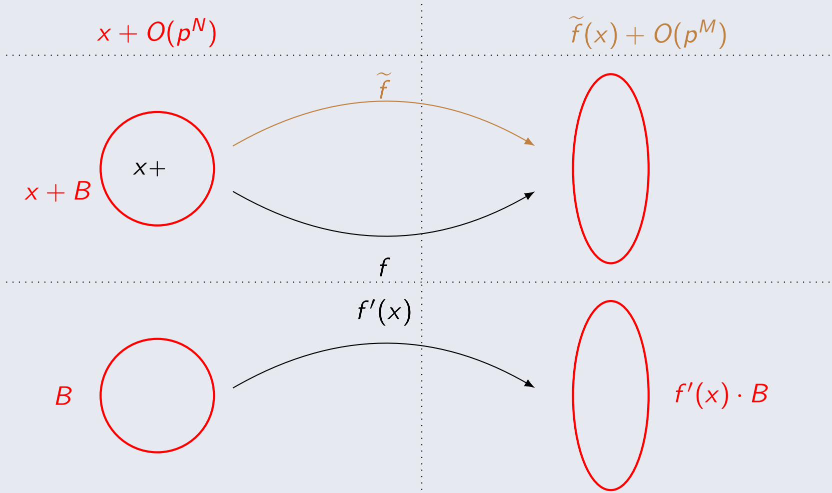

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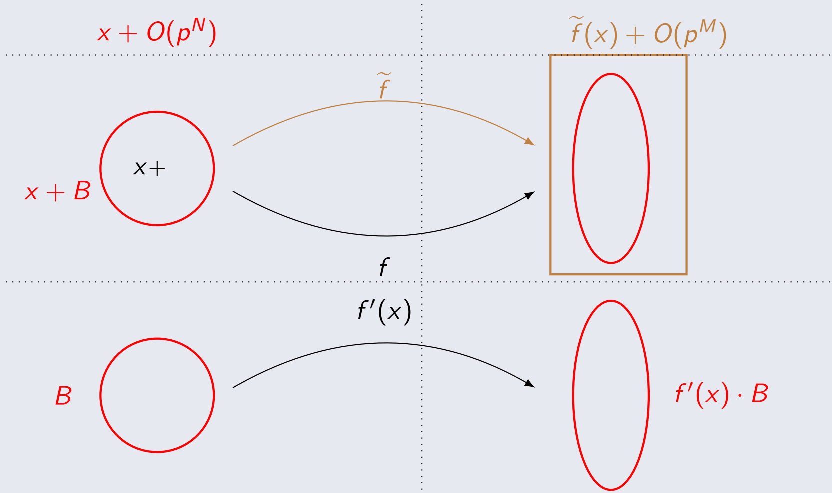

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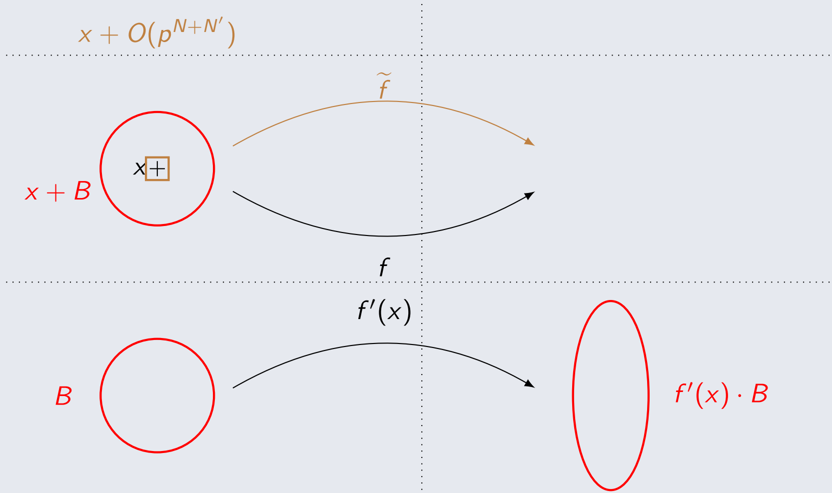

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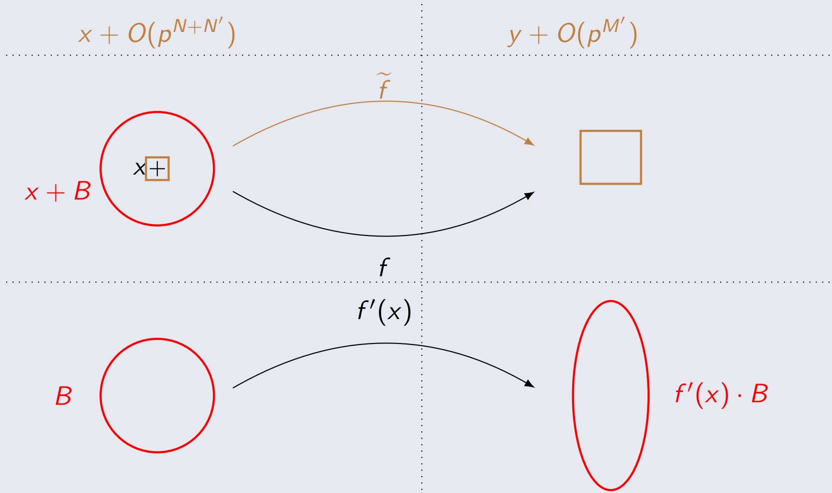

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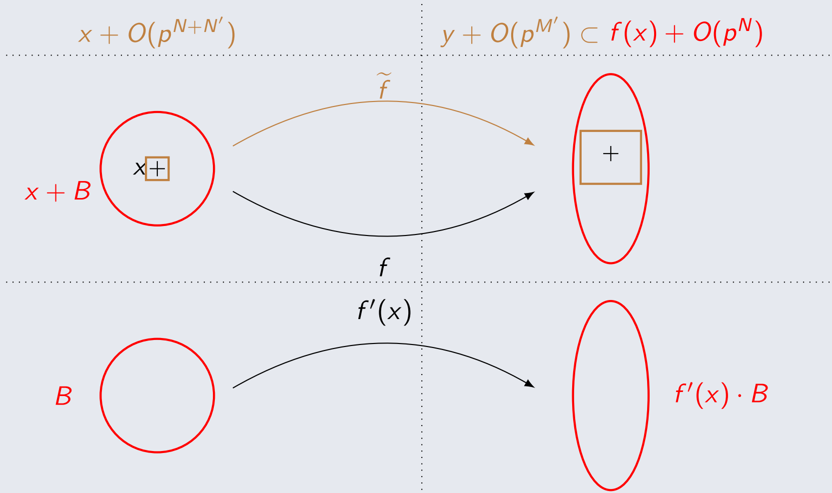

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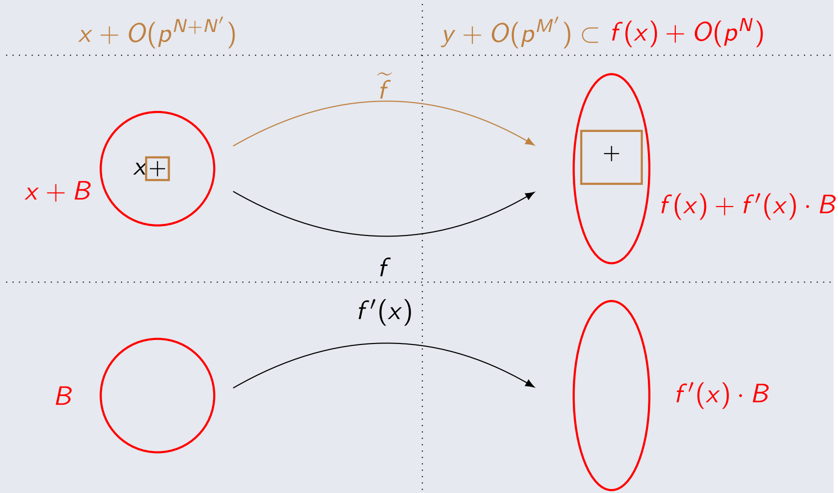


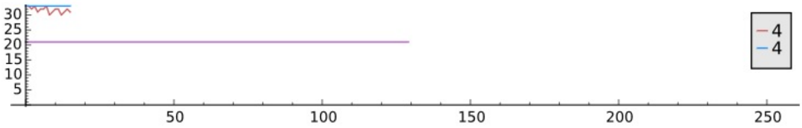
Buffer method

Proposition

One can use the following Newton scheme with lift:

- $\boxed{Lift}$ u_l to $O(p^{N+m})$,
- $a + O(x^{2^l}) \leftarrow u_l \times u_l'^{-1} + O(p^{N+m}, x^{2^l})$.
- $u_{l+1} \leftarrow u_l + \frac{u_l'}{a} \int (a \times u_l - u_l') \frac{a}{u_l'} + O(x^{2^{l+1}+1})$.

Numerical experiments



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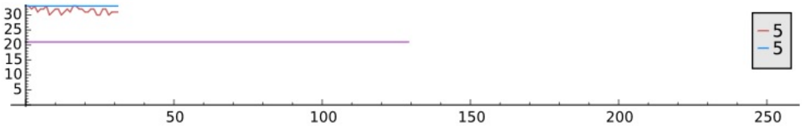

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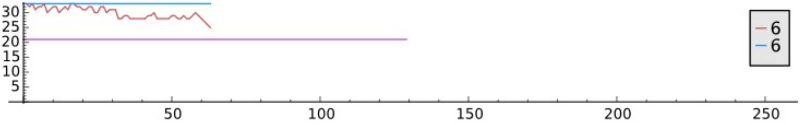

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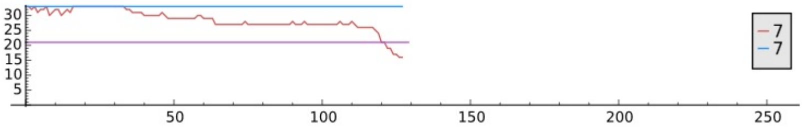

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Comparison of the methods

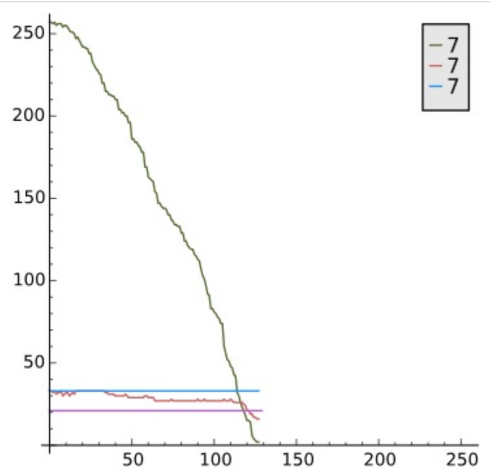

Figure: Naïve versus liftings

More differential equations

Generalization

This approach can be generalized to:

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- Differential equations to compute normalized isogenies between elliptic curves :

$$y'^2 = g(x) \times h(y(x)).$$

  Cf Lercier, Sirvent On elkies subgroups of l-torsion points in elliptic curves defined over a finite field.

To sum up

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- Implement automatic computation of differentials, within the computation.
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- More on differential equations.

References

Over real precision

- G.H.Golub, C.F.Van Loan Matrix Computation (John Hopkins University Press).

Over p-adic analysis

- P.Schneider p-adic Lie groups. (Springer).

Over these p-adic differential equations

- Bostan, González-Vega, Perdry, Schost From Newton sums to coefficients: complexity issues in characteristic p. (MEGA 05).

Over p-adic precision

- X.Caruso, D.Roe, T.Vaccon Tracking p-adic precision, ANTS XI.