Sparse Polynomial Systems With Many Positive Solutions From Bipartite Simplicial Complexes

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Real solutions of multivariate polynomial systems are ubiquitous in many applications of mathematics, as they often contain relevant information. Positive solutions are often of special interest, for instance when the variables represent physical quantities (*e.g.* in robotics), or if they represent probabilities (*e.g.* in algebraic statistics). Let $f_1, \ldots, f_n \in \mathbb{R}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ be Laurent polynomials. A real solution of $f_1(X_1, \ldots, X_n) = \cdots = f_n(X_1, \ldots, X_n) = 0$ is called positive if all its coordinates are positive. It is non-degenerate if the Jacobian matrix of the system is invertible at the solution. We focus on the following problem:

> Given an integer $n \in \mathbb{N}$ and a finite set $\mathcal{M}$ of Laurent monomials in n variables, construct n polynomials in $\mathbb{R}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ which involve only monomials in $\mathcal{M}$ and such that the system
>
> $$f_1(X_1, \ldots, X_n) = \cdots = f_n(X_1, \ldots, X_n) = 0$$
>
> has many non-degenerate positive solutions.

Our approach is related to a result by Sturmfels [4] which states that if the point configuration given by the exponent vectors of $\mathcal{M}$ admits a regular unimodular triangulation, then there exists a system with support $\mathcal{M}$ whose complex solutions with non-zero coordinate are all real. This is also related to Viro's theorem [6], to its extensions for complete intersections [2, 5] and to constructions of mixed systems based on signed Newton polytopes [3].

We propose a construction whose starting point is to consider a regular simplicial complex in $\mathbb{R}^n$ whose vertices are the exponent vectors of the monomials in $\mathcal{M}$. Next, the goal is to assign a vector in $\mathbb{R}^n$ to each vertex of the simplicial complex under sign constraints on some maximal minors of the matrix recording these vectors. From such an assignment, we construct a polynomial system whose number of positive and non-degenerate solutions is bounded below by the number of simplices in the simplicial complex. Computing this assignment boils down to a positive variant of the low-rank completion problem or, equivalently, to the problem of realizability of some oriented matroids.

We illustrate the construction with various examples and we identify, among classical families, finite sets of monomials $\mathcal{M}$ admitting *maximally positive systems*, *i.e.* systems with support $\mathcal{M}$ all complex toric solutions of which are positive and non-degenerate. These families provide evidence in favor of a conjecture by Bihan [1], which gives a necessary condition on such sets of monomials.

Finally, we discuss the construction of polynomial systems with many positive solutions when only the cardinality of $\mathcal{M}$ is prescribed.

Bibliographie

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