ALGORITHMS FOR D-FINITE FUNCTIONS

MANUEL KAUERS *

ABSTRACT. D-finite functions play an important role in the part of computer algebra concerned with algorithms for special functions. They are interesting both from a computational perspective as well as from the perspective of applications. We give an overview over the main properties and the classical algorithms for D-finite functions.

1. INTRODUCTION

A function is called *D-finite* if it is a solution of a linear differential equation with polynomial coefficients,

$$p_0(x)f(x) + p_1(x)f'(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

The letter "D" in the term D-finite stands for "differentiably", and the "finite" refers to the requirement that the order r of the equation should be finite. The notion was introduced by Stanley in 1980 [74], although many features of D-finite functions have been known for a long time without that these functions accquired a special name. A different way of stating the definition is as follows. A function f is called D-finite if the vector space generated by f and all its derivatives over the field of rational functions in x has a finite dimension:

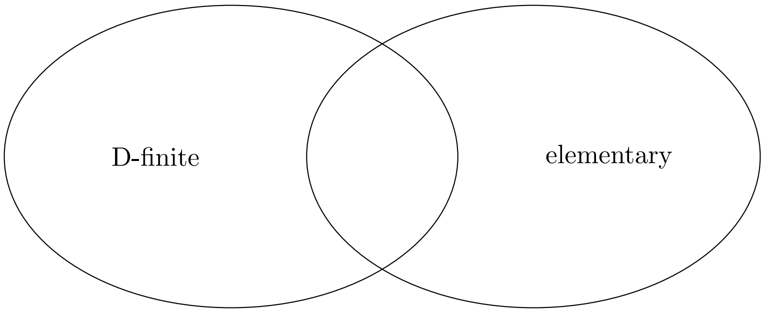
$$\mathbb{C}(x)f + \mathbb{C}(x)f' + \cdots = \mathbb{C}(x)f + \cdots + \mathbb{C}(x)f^{(r-1)}.$$

It is clear that this is the case if and only if f is the solution of a linear differential equation with polynomial coefficients.

This latter characterization allows the formulation of algorithms based on linear algebra for operating with D-finite functions. The first characterization is more user-friendly. For example, many classical special functions appearing in physics, engineering, statistics, combinatorics, etc. are solutions of linear differential equations with polynomial coefficients, and these equations together with a suitable set of initial values may be used as a definition for these functions. Classical [4] as well as modern [64] handbooks contain these equations, along with all sorts of other interesting information about these functions. An even more modern approach is the Dynamic Dictionary of Mathematical Functions (DDMF) [11], in which only the defining equations are stored in a database (or supplied by a user), and all the other information about the functions defined by these equations is generated on demand using computer algebra.

Besides D-finite functions, there are of course other classes of functions for which computer algebra algorithms are available. Another well-known class is the class of elementary functions, i.e., the set of all the functions which can be written in terms of an expression composed of exp, log, $+$, $-$, $/$, $\cdot$, and algebraic functions. These expressions are sometimes called "closed form expressions". Some but not all elementary functions are D-finite, and some but not all D-finite functions are elementary.

* Supported by the Austrian FWF grant Y464-N18.



While elementary functions can by definition always been written in terms of an explicit expression $f(x) = \cdots$, this is not always the case for D-finite functions. Instead, D-finite functions are specified implicitly through the differential equation they satisfy, plus an appropriate number of initial values. The situation is similar to the treatment of algebraic numbers: some but not all of them can be written in terms of nested radical expressions ("closed forms"). Algorithms operating on radical expressions therefore do not cover all algebraic numbers. In order to cover them all, we can use the minimal polynomial as a data structure, perhaps combined with some additional data for specifying which of the (finitely many) roots of the polynomial is meant.

If we are given a function, how can we decide whether it is D-finite? This is a good question, but there is no good answer to it. (Consider for comparison the analogous question for numbers: given a real number, how do you decide whether it is transcendental?) In particular, there cannot be an algorithmic answer to this question. The input of any algorithm necessarily needs to be some finite amount of data, and since there are uncountably many functions, no data structure can exist which allows to represent every of them in finite terms. In other words: there is no algorithm which takes an "arbitrary" complex function as input and then does something with it. There is not even an algorithm that takes an "arbitrary" complex number as input. The choice of a particular way of representing functions or numbers in finite terms necessarily restricts the consideration to a certain class of functions or numbers—those which admit such a representation.

If a function f is D-finite (or suspected to be so), but we do not know the differential equation that it satisfies, then it really depends on what we know about the function. For example, if we know that the function is the product of two D-finite functions for which we do know equations, then we can use an algorithm described below to construct an equation for f from the known equations of its factors. If we know that the function has a common name, such as the Bessel function, then we can look up some table and see if it contains the desired equation. Third, if whatever we know about f allows us to calculate the first terms of the series expansion of f at the origin, then we can use this data to search empirically for potential equations for f.

This technique, also known as "guessing", works as follows. Suppose that $f(x) = f_0 + f_1 x + \cdots + f_N x^N + \mathrm{O}(x^{N+1})$ for some known $f_0, \ldots, f_N$. Choose some numbers r and d such that $(r+1)(d+2) < N$. We want to test for possible equations of the form

$$p_0(x) f(x) + \cdots + p_r(x) f^{(r)}(x) = 0$$

where $p_0, \ldots, p_r$ are polynomials of degree at most d. A necessary condition is

$$p_0(x) f(x) + \cdots + p_r(x) f^{(r)}(x) = \mathrm{O}(x^{N-r}).$$

Since the first N terms in the expansion of f are known, we also know the first $N-i$ terms in the expansion of the ith derivative $f^{(i)}$, and so we know the first $N-r$

terms of all the functions $f, f', \ldots, f^{(r)}$. We want to find the polynomials $p_i(x)$, i.e., writing them in the form $p_i(x) = p_{i,0} + p_{i,1}x + \cdots + p_{i,d}x^d$, we want to know their coefficients $p_{i,j}$. If we equate the first $N-r$ coefficients of $p_0(x)f(x) + \cdots + p_r(x)f^{(r)}(x)$ to zero, we get $N-r$ linear constraints on the unknown coefficients $p_{i,j}$. Since there are $(r+1)(d+1)$ many unknowns and $N-r$ equations and we have chosen r and d in such a way that $(r+1)(d+2) = (r+1)(d+1) + r < N$, then this is an overdetermined linear system. We do not expect that such a system has a solution by accident, but by construction it must have all the valid equations for f of order at most r and degree at most d among its solutions. So if the system happens to have a solution, it is a fair guess that this solution corresponds to a valid equation for f.

As an example, consider $f(x) = \frac{1}{1+\sqrt{1-x}}$. Is this a D-finite function? Let's try to find a candidate equation with $r = d = 2$, i.e., an equation of the form

$$(p_{0,0}+p_{0,1}x+p_{0,2}x^2)f(x)+(p_{1,0}+p_{1,1}x+p_{1,2}x^2)f'(x)+(p_{2,0}+p_{2,1}x+p_{2,2}x^2)f''(x) = 0.$$

In order to ensure that we get an overdetermined system, we need $N > (r+1)(d+2) = 12$ terms, let's take $N = 15$:

$$\begin{aligned} f(x) = {} & \tfrac{1}{2} + \tfrac{1}{8}x + \tfrac{1}{16}x^2 + \tfrac{5}{128}x^3 + \tfrac{7}{256}x^4 + \tfrac{21}{1024}x^5 + \tfrac{33}{2048}x^6 \\ & + \tfrac{429}{32768}x^7 + \tfrac{715}{65536}x^8 + \tfrac{2431}{262144}x^9 + \tfrac{4199}{524288}x^{10} \\ & + \tfrac{29393}{4194304}x^{11} + \tfrac{52003}{8388608}x^{12} + \tfrac{185725}{33554432}x^{13} \\ & + \tfrac{334305}{67108864}x^{14} + \tfrac{9694845}{2147483648}x^{15} + \mathrm{O}(x^{16}). \end{aligned}$$

Plugging this expansion into the equation template and equating coefficients of $x^0, x^1, \ldots, x^{13}$ to zero yields the homogeneous linear system

$$\begin{pmatrix}
\frac{1}{2} & 0 & 0 & \frac{1}{8} & 0 & 0 & \frac{1}{8} & 0 & 0 \\
\frac{1}{8} & \frac{1}{2} & 0 & \frac{1}{8} & \frac{1}{8} & 0 & \frac{15}{64} & \frac{1}{8} & 0 \\
\frac{1}{16} & \frac{1}{8} & \frac{1}{2} & \frac{15}{128} & \frac{1}{8} & \frac{1}{8} & \frac{21}{64} & \frac{15}{64} & \frac{1}{8} \\
\frac{5}{128} & \frac{1}{16} & \frac{1}{8} & \frac{7}{64} & \frac{15}{128} & \frac{1}{8} & \frac{105}{256} & \frac{21}{64} & \frac{15}{64} \\
\frac{7}{256} & \frac{5}{128} & \frac{1}{16} & \frac{105}{1024} & \frac{7}{64} & \frac{15}{128} & \frac{495}{1024} & \frac{105}{256} & \frac{21}{64} \\
\frac{21}{1024} & \frac{7}{256} & \frac{5}{128} & \frac{99}{1024} & \frac{105}{1024} & \frac{7}{64} & \frac{9009}{16384} & \frac{495}{1024} & \frac{105}{256} \\
\frac{33}{2048} & \frac{21}{1024} & \frac{7}{256} & \frac{3003}{32768} & \frac{99}{1024} & \frac{105}{1024} & \frac{5005}{8192} & \frac{9009}{16384} & \frac{495}{1024} \\
\frac{429}{32768} & \frac{33}{2048} & \frac{21}{1024} & \frac{715}{8192} & \frac{3003}{32768} & \frac{99}{1024} & \frac{21879}{32768} & \frac{5005}{8192} & \frac{9009}{16384} \\
\frac{715}{65536} & \frac{429}{32768} & \frac{33}{2048} & \frac{21879}{262144} & \frac{715}{8192} & \frac{3003}{32768} & \frac{188955}{262144} & \frac{21879}{32768} & \frac{5005}{8192} \\
\frac{2431}{262144} & \frac{715}{65536} & \frac{429}{32768} & \frac{20995}{262144} & \frac{21879}{262144} & \frac{715}{8192} & \frac{1616615}{2097152} & \frac{188955}{262144} & \frac{21879}{32768} \\
\frac{4199}{524288} & \frac{2431}{262144} & \frac{715}{65536} & \frac{323323}{4194304} & \frac{20995}{262144} & \frac{21879}{262144} & \frac{1716099}{2097152} & \frac{1616615}{2097152} & \frac{188955}{262144} \\
\frac{29393}{4194304} & \frac{4199}{524288} & \frac{2431}{262144} & \frac{156009}{2097152} & \frac{323323}{4194304} & \frac{20995}{262144} & \frac{7243275}{8388608} & \frac{1716099}{2097152} & \frac{1616615}{2097152} \\
\frac{52003}{8388608} & \frac{29393}{4194304} & \frac{4199}{524288} & \frac{2414425}{33554432} & \frac{156009}{2097152} & \frac{323323}{4194304} & \frac{30421755}{33554432} & \frac{7243275}{8388608} & \frac{1716099}{2097152} \\
\frac{185725}{33554432} & \frac{52003}{8388608} & \frac{29393}{4194304} & \frac{2340135}{33554432} & \frac{2414425}{33554432} & \frac{156009}{2097152} & \frac{1017958725}{1073741824} & \frac{30421755}{33554432} & \frac{7243275}{8388608}
\end{pmatrix}.$$

Each vector $(p_{0,0}, p_{0,1}, p_{0,2}, p_{1,0}, p_{1,1}, p_{1,2}, p_{2,0}, p_{2,1}, p_{2,2})$ in the (right) kernel of this matrix corresponds to a potential differential equation for f. Indeed, the kernel turns out to be nontrivial, and we find the equation

$$f(x) + (5x-4)f'(x) + (2x^2-2x)f''(x) = 0.$$

To prove that this equation is correct, it suffices to plug the known closed form expression for f into the left hand side and check that it simplifies to zero. (It does.)

Guessing is a powerful technique because it is simple, it requires almost no knowledge about the function, and it can be easily adapted to other situations. It is among the most popular functionalities of software packages for D-finite functions. Modern implementations do not use the naive algorithm sketched above, but are based on Hermite-Pade approximation [10] as well as a technique called "trading order against degree" [49, 27, 54], and they use homomorphic images [86] to speed up the computation. These implementations have no trouble finding also extremely large equations for which N must be in the range of 10000 or so.

The notion of D-finiteness is more general than suggested above. We may allow other operations instead of the derivation, for example the shift operators $f(x) \mapsto f(x+1)$. Then, instead of linear differential equations with polynomial coefficients we have linear recurrence equations with polynomial coefficients. Also the q-shift $f(x) \mapsto f(qx)$ (where q is some constant different from 1) is sometimes of interest. Many of the algorithms for D-finite functions extend to these cases, and in order to formulate them in a uniform manner, it is convenient to employ the general notion of Ore algebras, which we will explain in Section 2.2 below.

Secondly, the notion of D-finiteness also extends to the case of multivariate functions. Several results generalize in a straightforward way to this case, but there are also some new aspects which do not show up in the univariate case. This survey is divided into two parts. In the first part, we discuss the univariate case, and in the second we consider the case of multivariate D-finite functions.

2. One Variable

2.1. **Examples.** Many interesting functions are D-finite. For example:

(1) $f(x) = x^2+3$ is D-finite. It satisfies the differential equation $2xf(x)-(x^2+3)f'(x) = 0$. In fact, every polynomial is D-finite. More generally, every rational function is D-finite.

(2) $f(x) = \frac{1}{1+\sqrt{x-1}}$ is D-finite, as we have seen above. In fact, every algebraic function is D-finite.

(3) f is called hyperexponential if $f'(x)/f(x)$ is a rational function. Obviously, all hyperexponential functions are D-finite.

(4) The hypergeometric function

$$f(x) = {}_2F_1\left(\begin{array}{cc} a & b \\ & c \end{array} \middle| \, x\right) = \sum_{n=0}^{\infty} \frac{a^{\overline{n}} b^{\overline{n}}}{c^{\overline{n}} n!} x^n,$$

where $u^{\overline{n}} = u(u+1)\cdots(u+n-1)$ denotes the rising factorial and a, b, c are constants, is D-finite. It satisfies the differential equation

$$abf(x) - (c - (a+b+1)x)f'(x) - x(x-1)f''(x) = 0.$$

More generally, the generalized hypergeometric function

$${}_pF_q\left(\begin{array}{cccc} a_1 & a_2 & \cdots & a_p \\ & b_1 & \cdots & b_q \end{array} \middle| \, x\right) = \sum_{n=0}^{\infty} \frac{a_1^{\overline{n}} a_2^{\overline{n}} \cdots a_p^{\overline{n}}}{b_1^{\overline{n}} \cdots b_q^{\overline{n}} n!} x^n$$

is D-finite. The class of (generalized) hypergeometric functions includes the classical orthogonal polynomials as well as Bessel functions and many other well-known special functions. So all these functions are in particular D-finite.

(5) The elementary function $\log(1+\sqrt{x}) + \exp(x)$ is neither algebraic nor hypergeometric, but it is D-finite. It satisfies the linear differential equation

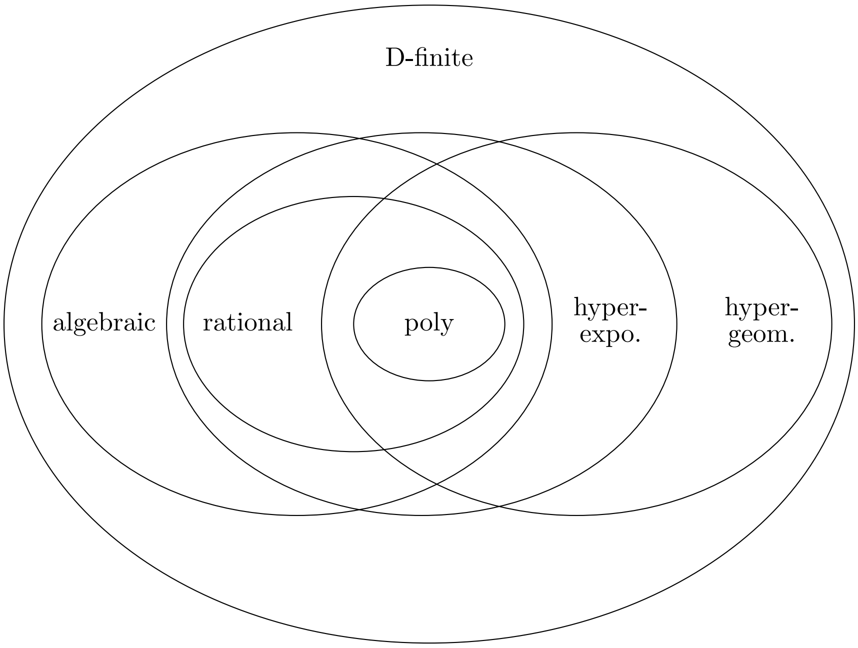
$$(-6x^2 - 27x - 15)f'(x) + (-14x^3 - 37x^2 + 42x + 15)f''(x)$$
$$+ (-4x^4 + 8x^3 + 53x^2 - 15x)f^{(3)}(x) + (4x^4 + 6x^3 - 10x^2)f^{(4)}(x) = 0$$

(6) The function $f(x)$ with

$$(1+2x+3x^2)f(x)+(4+5x+6x^2)f'(x)$$
$$+(7+8x+9x^2)f''(x)+(10+11x+12x^2)f'''(x)=0$$

and $f(0)=f'(0)=f''(0)=1$ is D-finite, but it is not elementary or hypergeometric.

(7) The function $\exp(\exp(x))$ is elementary, but not D-finite. The Gamma function $\Gamma(x)$ is neither elementary nor D-finite.



When other functional equations instead of differential equations are used, some care must be applied to avoid confusion. A particular function may be D-finite with respect to some operation but not with respect to another. For example, the gamma function $\Gamma(x)$ does not satisfy any linear differential equation with polynomial coefficients, and is therefore not D-finite. However, it does satisfy the linear recurrence equation $xf(x)-f(x+1)=0$, so it is D-finite *with respect to the shift operator.* For most functions, it will be clear from the context which operation is meant, and then we do not need to state it. Here are some objects which are D-finite with respect to the shift operator:

(1) $f(x)=x^3+1$ satisfies the recurrence $((x+1)^3+1)f(x)-(x^3+1)f(x+1)=0$. Every polynomial, and in fact every rational function is D-finite not only with respect to differentiation but also with respect to shift.

(2) $f(x)=\Gamma(x)$ is D-finite with respect to shift. More generally, a function f is called a hypergeometric term if $f(x+1)/f(x)$ is a rational function. Every hypergeometric term is D-finite with respect to shift.

(3) The sequence of Fibonacci numbers is D-finite with respect to shift. It satisfies by definition the recurrence $f(x)+f(x+1)-f(x+2)=0$. More generally, every sequence which satisfies a linear recurrence equation with constant coefficients in particular satisfies an equation with polynomial coefficients and is therefore D-finite.

(4) The sequence of Apery numbers $f(n) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2$ is D-finite. It satisfies the recurrence

$$(n+1)^3 f(n) - (2n+3)\left(17n^2 + 51n + 39\right) f(n+1) + (n+2)^3 f(n+2) = 0.$$

It is not obvious that the sum satisfies this recurrence, but it can be checked with creative telescoping discussed below.

(5) The sequence of Legendre polynomials $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$ is D-finite with respect to shift when x is a fixed parameter and n is the variable. It satisfies the recurrence

$$(n+1)P_n(x) - (2n+3)xP_{n+1}(x) + (n+2)P_{n+2}(x) = 0.$$

It is also D-finite with respect to the derivation when we view n as a fixed parameter and let x be the variable. The differential equation is

$$n(1+n)P_n(x) - 2xP_n'(x) + (1-x)(1+x)P_n''(x) = 0.$$

Similar equations hold for all other classical families of orthogonal polynomials.

(6) The sequences $\sqrt{n}$, $\log(n)$, 2^{2^n} and $\frac{1}{1+n!}$ are not D-finite with respect to shift [40, 39].

Finally, here are some quantities which are D-finite with respect to the q-shift. In all these examples, q may be any quantity different from 1. Typically, it is most convenient to assume that q is transcendental over $\mathbb{C}$, or at least that q is not a root of unity. There are two ways to interpret the solution of a q-shift equation. Either x is considered as a continuous variable which the operation scales by a factor of q (then the q-shift is also called q-dilation). Or we can view x as a symbol that represents the power q^n, and then the q-shift amounts to the usual shift that sends n to $n+1$. In our examples, we will restrict to the latter interpretation.

(1) $f(n) := [n]_q := \frac{q^n-1}{q-1}$ is D-finite with respect to the q-shift, because it satisfies $(q^n-1)f(n+1) - (q\,q^n-1)f(n) = 0$. More generally, if $r(x)$ is any rational function, then $r([n]_q)$ is D-finite with respect to the q-shift.

(2) $f(n) = 2^n$ is D-finite with respect to the q-shift. In fact, every linear recurrence with constant coefficients is in particular a linear recurrence with coefficients that are polynomials in q^n, and therefore any solution of a linear recurrence with constant coefficients is in particular D-finite with respect to the q-shift.

(3) $f(n) := [n]_q! := [n]_q[n-1]_q \cdots [1]_q$ is D-finite with respect to the q-shift. It satisfies the recurrence $(1-q\,q^n)f(n) + (q-1)f(n+1) = 0$.

(4) The basic hypergeometric function

$$f(x) := {}_2\phi_1\left(\begin{array}{cc} a & b \\ & c \end{array}\middle|\, x\right) := \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(c;q)_n (q;q)_n} x^n,$$

where $(u;q)_n := (1-u)(1-qu)\cdots(1-q^{n-1}u)$ denotes the q-Pochhammer symbol, is D-finite with respect to the q-dilation. It satisfies the equation

$$(x-1)q\,f(x) + (c+q-aqx-bqx)f(qx) - (c-abqx)f(q^2x) = 0.$$

2.2. **Algebraic Setup.** In order to treat all the various flavors of D-finiteness in a uniform way, it is convenient to adopt the viewpoint of operators. Operators are elements of a certain algebra that acts in a certain way on a function space. To make this precise, we use the notion of Ore algebras [66].

Definition 1. *Let A be an integral domain, $\sigma\colon A \to A$ an endomorphism, $\delta\colon A \to A$ a σ-derivation, i.e., an A-linear map with the property that $\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$ for all $a, b \in A$.*

Let $A[\partial]$ be the set of all univariate polynomials in the indeterminate ∂ with coefficients in A. Addition in $A[\partial]$ is defined coefficient-wise (as usual), and multiplication is defined via the commutation rule $\partial a = \sigma(a)\partial + \delta(a)$ for all $a \in A$.

Then $\mathbb{A} := (A[\partial], \sigma, \delta)$ is called an Ore algebra over A.

The prototypical example of an Ore algebra is $\mathbb{C}[x][\partial]$ with σ the identity map and $\delta = \frac{d}{dx}$ the partial derivative. This particular algebra is also known as a Weyl-algebra. Thanks to the commutation rule $\partial a = \sigma(a)\partial + \delta(a)$, elements of an Ore algebra can always be written as a sum of terms of the form $a\partial^i$, where a is an element of the ground domain A. For example, in the Weyl algebra we have

$$\begin{aligned}
&(a + b\partial + c\partial^2)(d + e\partial)\\
&= a(d + e\partial) + b\partial(d + e\partial) + c\partial^2(d + e\partial)\\
&= ad + ae\partial + b\partial d + b\partial e\partial + c\partial^2 d + c\partial^2 e\partial\\
&= ad + ae\partial + b(d\partial + d') + b(e\partial + e')\partial + c\partial(d\partial + d') + c\partial(e\partial + e')\partial\\
&= ae + ae\partial + bd\partial + bd' + be\partial^2 + be'\partial\\
&\quad + c(d\partial + d')\partial + c(d'\partial + d'') + c(e\partial + e')\partial^2 + c(e'\partial + e'')\partial\\
&= (ae + bd' + cd'') + (ae + bd + be' + 2cd' + ce'')\partial + (be + cd + 2ce')\partial^2 + ce\partial^3.
\end{aligned}$$

The elements of this algebra $\mathbb{C}[x][\partial]$ are called differential operators. For the ground domain, we can also choose the rational function field $\mathbb{C}(x)$ instead of the polynomial ring $\mathbb{C}[x]$. Of course, we can also take other fields in place of $\mathbb{C}$.

Another prominent example for an Ore algebra is $\mathbb{C}[x][\partial]$ with $\sigma\colon \mathbb{C}[x] \to \mathbb{C}[x]$ defined via $p(x) \mapsto p(x+1)$ and $\delta = 0$. In this case the commutation rule reads $\partial x = (x+1)\partial$. The elements of this algebra are called recurrence operators.

As a third example, consider $\mathbb{C}(q)[x][\partial]$ with the commutation rule $\partial x = qx\partial$, i.e., $\sigma\colon \mathbb{C}(q)[x] \to \mathbb{C}(q)[x]$ with $p(x) \mapsto p(qx)$ and $\delta = 0$. In this case the elements of the algebra are called q-recurrence operators.

Elements of Ore algebras are used to describe functions. In order to make this precise, we consider "function spaces" F as left-$A[\partial]$-modules, i.e., there is an action $A[\partial] \times F \to F$ which maps an operator $L \in A[\partial]$ and a function $f \in F$ to the function $L \cdot f \in F$. For example, the Ore algebra $\mathbb{C}(x)[\partial]$ of linear differential operators acts in a natural way on the field $\mathcal{M}_{\mathbb{C}}$ of meromorphic functions or on the field $\mathbb{C}((x))$ of formal Laurent series. The element x acts via multiplication ($x \cdot f = xf$) and the generator ∂ acts as derivation ($\partial \cdot f = f'$). The commutation rule of the algebra is chosen in such a way that the multiplication of the algebra becomes compatible with the action on the function space, i.e., so that we have $(ML) \cdot f = M \cdot (L \cdot f)$ for all $M, L \in \mathbb{C}(x)[\partial]$ and all $f \in F$. For example, $f + xf' = (xf)' = \partial \cdot (xf) = (\partial x) \cdot f = (x\partial + 1) \cdot f = (x\partial) \cdot f + (1 \cdot f) = xf' + f$.

The algebra of recurrence operators also acts in a natural way on the field $\mathcal{M}_{\mathbb{C}}$, as well as on domains of sequences such as $\mathbb{C}^{\mathbb{N}}$ or $\mathbb{C}^{\mathbb{Z}}$. The element x acts again via multiplication, and the generator ∂ corresponds to the shift $f(x) \mapsto f(x+1)$. Again, the commutation rule ensures compatibility of the algebra multiplication and the action: $(x+1)f(x+1) = \partial \cdot (xf(x)) = (\partial x) \cdot f(x) = ((x+1)\partial) \cdot f(x) = (x+1) \cdot (\partial \cdot f(x)) = (x+1) \cdot f(x+1) = (x+1)f(x+1)$.

Let $A[\partial]$ be an Ore algebra which acts on F. For a fixed $f \in F$, we call

$$\operatorname{ann}(f) := \{\, L \in A[\partial] : L \cdot f = 0 \,\}$$

the *annihilator* of f. It is easy to see that this is a left ideal of $A[\partial]$. Furthermore, for a fixed $L \in A[\partial]$, we call

$$V(L) := \{\, f \in F : L \cdot f = 0 \,\}$$

the *solution space* of f. An element $a \in A$ is called a *constant* if $\sigma(a) = a$ and $\delta(a) = 0$. The set C of constants forms a subring of A. The solution space is a C-module.

Examples:

- If $\mathbb{C}(x)[\partial]$ denotes the algebra of recurrence or differential operators, its field of constants is $\mathbb{C}$.
- Let $\mathbb{C}[x][\partial]$ be the algebra of differential operators, and let it act on the ring $F = \mathbb{C}[[x]]$ of formal power series. Then we have, for example,

$$\begin{aligned} \operatorname{ann}(\exp(x)) &= \mathbb{C}(x)[\partial]\ (\partial - 1) \\ \operatorname{ann}(\sqrt{1-x}) &= \mathbb{C}(x)[\partial]\ (2(1-x)\partial - 1) \\ \operatorname{ann}(0) &= \mathbb{C}(x)[\partial]\ 1 \\ V(\partial - 1) &= \mathbb{C}\ \exp(x) \\ V((1+x)\partial^2 + \partial) &= \mathbb{C}\ 1 + \mathbb{C}\ \log(1+x). \end{aligned}$$

- Let $\mathbb{C}(x)[\partial]$ be the algebra of recurrence operators, and let it act on the ring $\mathbb{C}^{\mathbb{Z}}$ of sequences. Then we have, for example,

$$\begin{aligned} \operatorname{ann}(2^x) &= \mathbb{C}(x)[\partial]\ (\partial - 2) \\ \operatorname{ann}(x^2) &= \mathbb{C}(x)[\partial]\ (x^2\partial - (x+1)^2) \\ \operatorname{ann}(\sqrt{x}) &= \{0\} \\ V(\partial - 1) &= \mathbb{C} \\ V(\partial^2 - 5\partial + 3) &= \mathbb{C}\ 2^x + \mathbb{C}\ 3^x. \end{aligned}$$

Definition 2. *Let $A[\partial]$ be an Ore algebra which acts on F. An element $f \in F$ is called D-finite (with respect to the action of $A[\partial]$ on F) if* $\operatorname{ann}(f) \neq \{0\}$.

In other words, $f \in F$ is D-finite if and only if there exists an operator $L \neq 0$ with $L \cdot f = 0$.

Despite the non-commutative multiplication, an Ore algebra $A[\partial]$ is not much different from a usual univariate polynomial ring $A[x]$. In particular, when A is a field, then we can do division with remainder in $A[\partial]$, either from the left or from the right. Right division is more useful in our setting. It says that when we have $L, M \in A[\partial]$ with $M \neq 0$, then there exists a unique pair $U, V \in A[\partial]$ with $\deg_\partial V < \deg_\partial M$ and $L = UM + V$. We call U the right quotient and V the right remainder of L upon division by M.

Suppose f is a D-finite function, say with $L \cdot f = 0$ for some operator L. Suppose further that M is an arbitrary other operator. Then $M = UL + V$ implies that $M \cdot f = (UL + V) \cdot f = (UL) \cdot f + V \cdot f = U \cdot (L \cdot f) + V \cdot f = U \cdot 0 + V \cdot f = V \cdot f$. Therefore, we have

$$\begin{aligned} A[\partial] \cdot f &= A\ f + A\ (\partial \cdot f) + \cdots + A\ (\partial^{\deg_\partial L - 1} \cdot f) \\ &\cong A[\partial]/\operatorname{ann}(f). \end{aligned}$$

The first line is an algebraic reformulation of the second definition stated at the beginning of the introduction: f is D-finite if and only if the vector space generated by f and its "derivatives" over A has finite dimension. The second line rephrases the homomorphism theorem. In applications, we want to say something about f, but the domain F in which f lives is not suitable for computations. When f is

D-finite, we can instead compute in $A[\partial]/\operatorname{ann}(f)$, which is a perfectly nice algebraic setting where computations can be done.

The option of doing division with remainder (still assuming that A is a field) also implies that there is a notion of a greatest common right divisor of two operators. The greatest common right divisor is unique up to left-multiplication by a nonzero element of A, and it can be computed by a straightforward adaption of the Eucidean algorithm. For operators M, L we have $V(\operatorname{gcrd}(M, L)) = V(M) \cap V(L)$. Compare this to the case of usual commutative polynomials: the roots of $\gcd(p, q)$ are precisely the common roots of p and q.

Still continuing to assume that A is a field, there is also a notion of a least common left multiple of $L, M \in A[\partial]$: this is an operator of smallest possible degree in ∂ which has both M and L as right-divisors. It is unique up to left-multiplication by a nonzero element of A, and its degree in ∂ is at most the sum of the degrees of M and L, and at least the maximum of their degrees. In the case of univariate commutative polynomials, the computation of least common left multiples can be reduced to the computation of a product and a greatest common divisor via the formula $\operatorname{lcm}(p, q) = pq/\gcd(p, q)$. Unfortunately, there there is no such formula in the commutative case. In particular, it is not true that $\operatorname{gcrd}(M, L) = 1$ implies that $\operatorname{lclm}(L, M)$ is equal to ML or LM.

There are several algorithms for computing the least common left multiple of two given operators $L, M \in A[\partial]$. One of them consists in making an ansatz with undetermined coefficients for a left multiple of L and a left multiple of M and to force them to be equal by equating coefficients. This yields a linear system of equations with coefficients in A. To be specific, let $r = \deg_\partial L$ denote the order of L and $s = \deg_\partial M$ denote the order of M. Then we want $u_0, \dots, u_s$, $v_0, \dots, v_r \in A$ such that

$$(u_0 + u_1\partial + \cdots + u_s\partial^s)L = (v_0 + v_1\partial + \cdots + v_r\partial^r)M.$$

Note that the unknown coefficients are already at the left, so when we commute the ∂^i with the coefficients of L and M in order to bring the above equation into the form

$$(\cdots) + (\cdots)\partial + \cdots + (\cdots)\partial^{r+s} = (\cdots) + (\cdots)\partial + \cdots + (\cdots)\partial^{r+s},$$

then all the coefficient expressions $(\cdots)$ will be linear combinations of the unknowns $u_0, \dots, u_s$, $v_0, \dots, v_r$ with explicit elements of A as coefficients. This system must have a nontrivial solution, because we have $(s+1)+(r+1) = r+s+2$ variables and $r + s + 1$ equations. If the solution space has dimension one, then we can extract the coefficients of the left multiplier $u_0 + u_1\partial + \cdots + u_s\partial^s$ from a basis element, and multiplying it with L gives the least common left multiple. (Alternatively, of course, we can also extract the coefficients of $v_0 + v_1\partial + \cdots + v_r\partial^r$ and multiply it to M, and this will give the same result.) If the dimension is greater than one, we can keep repeating the procedure with an ansatz of lower and lower order in the ansatz for the multipliers, until the solution space has dimension one.

For more efficient algorithms for computing least common left multiples, see [15]. Improved algorithms for greatest common right divisors of Ore polynomials are discussed in [45, 46].

2.3. **Closure Properties.** If $L, M \in A[\partial]$ are two operators, f is a solution of L and g is a solution of M, then both f and g are solutions of $\operatorname{lclm}(L, M)$. Because $\operatorname{lclm}(L, M) = UL$ for some U implies that $(UL) \cdot f = U \cdot (L \cdot f) = U \cdot 0 = 0$ and $\operatorname{lclm}(L, M) = VM$ for some V implies that $(VM) \cdot g = 0$. Since the solution space of Ore operators is a vector space over the constant field, it follows that $f + g$ is also a solution of $\operatorname{lclm}(L, M)$.

We have thus shown that the sum of two D-finite functions is again D-finite. Moreover, the argument is algorithmic in the sense that there is an algorithm which constructs from given annihilating operators for f and g an annihilating operator for $f+g$.

The class of D-finite functions is also closed under multiplication, if the function space F on which the Ore algebra acts admits a multiplication which is compatible with the action of the algebra. To keep things simple, let us consider the differential case only. In this case, we have $\partial \cdot (fg) = (\partial \cdot f)\, g + f\,(\partial \cdot g)$ for all functions f, g. By induction, every derivative $\partial^m \cdot (fg)$ can be rewritten as a linear combination of terms of the form $(\partial^i \cdot f)(\partial^j \cdot g)$. If f and g are D-finite, then all the $\partial^i \cdot f$ for $i = 0, 1, 2, \dots$ belong to some finite-dimensional vector space, and so do all the $\partial^j \cdot g$ for $j = 0, 1, 2, \dots$. If the respective dimensions are r and s, then all the terms $(\partial^i \cdot f)(\partial^j \cdot g)$ for $i, j = 0, 1, \dots$ belong to some vector space of dimension at most rs. In this space, any choice of $rs+1$ elements must be linearly dependent. In particular, $fg, \partial\cdot(fg), \dots, \partial^{rs}\cdot(fg)$ must be linearly dependent, and the dependence gives rise to a differential equation satisfied by fg.

The argument translates into the following algorithm.

INPUT: $L, M \in A[\partial]$, where A is a field and $A[\partial]$ is the Ore algebra with $\sigma(p) = p$, $\delta(p) = p'$

OUTPUT: $L \otimes M \in A[\partial]$, an operator with the property that for all $f \in V(L)$ and all $g \in V(M)$ we have $fg \in V(L \otimes M)$.

1 Let $r = \deg_\partial L$, $s = \deg_\partial M$.

2 Let S be a matrix over A with rs rows and $rs+1$ columns

3 for $i = 0, \dots, rs$ do

4 Set the ith column of S to the coefficient vector of

$$\sum_{j=0}^{i} \operatorname{rrem}(\partial^j, L) \otimes \operatorname{rrem}(\partial^{i-j}, M).$$

By $\operatorname{rrem}(U, V)$ we mean (the coefficient vector in $A^{\deg_\partial V}$) of the remainder of U upon right division by V, and by $\otimes$ we mean the tensor product $(x_1, \dots, x_n) \otimes (y_1, \dots, y_m) = (x_1y_1, x_1y_2, \dots, x_ny_{m-1}, x_ny_m)$.

5 determine a nonzero vector $z = (z_0, \dots, z_{rs}) \in A^{rs+1}$ such that $Sz = 0$. (Such a vector exists.)

6 return $z_0 + z_1\partial + \dots + z_{rs}\partial^{rs}$.

The operator $L \otimes M$ computed by this algorithm is called the symmetric product of L and M. For other Ore algebras, only the expression in line 4 has to be changed. For example, in the shift case we have the product rule

$$\partial \cdot (f(x)g(x)) = f(x+1)g(x+1) = (\partial \cdot f(x))(\partial \cdot g(x)),$$

so the sum has to be replaced by $\operatorname{rrem}(\partial^i, L) \otimes \operatorname{rrem}(\partial^i, L)$ in this case.

The class of D-finite functions is also closed under application of ∂: when f is D-finite, then so is $\partial \cdot f$. This is clear, because for f to be D-finite means that f and all its derivatives $\partial^i \cdot f$ for $i = 0, 1, 2, \dots$ live in some finite dimensional vector space over the ground field A, and therefore obviously also $\partial \cdot f$ and all its derivatives $\partial^i \cdot (\partial \cdot f) = \partial^{i+1} \cdot f$ for $i = 0, 1, 2, \dots$ live in some finite dimensional vector space over A, so $\partial \cdot f$ is D-finite again.

More generally, when $M \in A[\partial]$ is any operator, not necessarily $M = \partial$, then for every D-finite function $f \in F$ also the function $M \cdot f$ is D-finite. The idea behind the algorithm for computing an annihilating operator for $M \cdot f$ from a given annihilating operator L of f is the same as before: compute the derivatives, write them in terms of a basis, and then find a linear relation among them.

INPUT: $L, M \in A[\partial]$
OUTPUT: $U \in A[\partial]$ such that for all $f \in F$ with $L \cdot f = 0$ we have $UM \cdot f = 0$.

```
Set r = deg_∂ L.
Let S be a matrix over A with r rows and r + 1 columns
for i = 0, ..., r do
   set the ith column of S to the coefficient vector in A^r of rrem(∂^i M, L).
determine a nonzero vector z = (z_0, ..., z_r) ∈ A^{r+1} such that Sz = 0. (Such a
vector exists.)
return z_0 + z_1 ∂ + ··· + z_r ∂^r.
```

As an example, let $\mathbb{Q}[x][\partial]$ be the algebra of recurrence operators, and consider the operator $L = \partial^2 - \partial - 1$. If $F = \mathbb{Q}^{\mathbb{Z}}$ is the space of sequences in $\mathbb{Q}$ and $\mathbb{Q}[x][\partial]$ acts on F in the natural way, then $V(L)$ contains the sequence $f = (F_n)_{n=0}^{\infty}$ of Fibonacci numbers. Let $M = (x+1)\partial^2 + (2x+1)\partial + (x-1)$ and $g = M \cdot f$. We want to find an operator U with $U \cdot g = 0$. Following the steps of the algorithm as outlined above, we compute

$$\begin{aligned} \operatorname{rrem}(M, L) &= (3x+2)\partial + 2x \\ \operatorname{rrem}(\partial M, L) &= (5x+7)\partial + (3x+5) \\ \operatorname{rrem}(\partial^2 M, L) &= (8x+20)\partial + (5x+12) \end{aligned}$$

A nontrivial solution of the linear system

$$\begin{pmatrix} 3x+2 & 5x+7 & 8x+20 \\ 2x & 3x+5 & 5x+12 \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix}$$

is $(-x^2+5x+16, -x^2+6x+24, x^2-7x-10)$. Consequently, we have

$$\big((x^2-7x-10)\partial^2 + (-x^2+6x+24)\partial + (-x^2+5x+16)\big) \cdot g = 0.$$

It follows directly from the algorithms (or rather from the proofs behind them) that when $L \cdot f = 0$ and $M \cdot g = 0$, then there is an operator which kills $f + g$ and has order at most $\deg_\partial L + \deg_\partial M$ as well as an operator which kills fg of order at most $(\deg_\partial L)(\deg_\partial M)$. Furthermore, for $U \cdot f$ for any $U \in A[\partial]$ there is an annihilating operator of order at most r. In the most common case when A is a rational function field in one variable, $A = C(x)$, it is also possible to give bounds on the degrees in x of the resulting operators. Furthermore, when $C = \mathbb{Q}$, it is also possible to formulate bounds on bitsize of the integers appearing in the output, and hence of the bitsize of the entire output operator. The formulas for these bounds are messy, and can be found in [52].

There are some additional closure properties which are special to particular Ore algebras.

- In the differential case, we have that when f is D-finite and g is algebraic (i.e., $P(x, g(x)) = 0$ for some nonzero bivariate polynomial P), then the composition $f \circ g$ is D-finite.
  We also have that $\int f$ is D-finite when f is.
- In the shift case, we have that when f is D-finite and g is integer-linear (i.e., $g(x) = ux + v$ for some specific integers $u, v \in \mathbb{Z}$), then the composition $f \circ g$ is D-finite.
  We also have that $\Sigma f = (\sum_{k=0}^{n} f_k)_{n=0}^{\infty}$ is D-finite when $f = (f_n)_{n=0}^{\infty}$ is.

- For formal power series $f(x) = \sum_{n=0}^{\infty} f_n x^n$, we have that $f(x)$ is D-finite with respect to the Ore algebra of differential operators if and only if the coefficient sequence $(f_n)_{n=0}^{\infty}$ is D-finite with respect to the Ore algebra of recurrence operators.

Of course, all these properties are algorithmic. For the first two items, the proofs and algorithms work very much like in the cases discussed above by finding a linear dependence among sufficiently many derivatives that are known to belong to some finite dimensional vector space. The third case is merely a rewriting based on the observation that $x\partial$ acts on $f(x)$ like n acts on $(f_n)_{n=0}^{\infty}$ and x acts on $f(x)$ like ∂^{-1} acts on $(f_n)_{n=0}^{\infty}$. (Some adjustment is needed because there are no negative powers of ∂ in an Ore algebra according to the definition that we use.)

Closure properties are useful for proving special function identities. As an example, let us prove the formula for the generating function of Legendre polynomials:

$$\sum_{n=0}^{\infty} P_n(x) z^n = \frac{1}{\sqrt{1 - 2xz + z^2}}$$

where the sequence $(P_n(x))_{n=0}^{\infty}$ is defined via the recurrence

$$(1+n)P_n(x) - (3+2n)x\,P_{n+1}(x) + (2+n)P_{n+2}(x) = 0$$

and the initial values $P_0(x) = 1$, $P_1(x) = x$. This is clearly a D-finite sequence with respect to n, when x is considered as a fixed constant. On the left hand side of the identity, we have the formal power series with $P_n(x)$ as coefficient sequence. Using a closure properties algorithm, we can translate the recurrence for $P_n(x)$ into a differential equation for this power series. One of several possible outputs is

$$(z^3 - 2xz^2 + z)f^{(3)}(z) + (7z^2 - 9xz + 2)f''(z) + (10z - 6x)f'(z) + 2f(z) = 0.$$

As a next step, we could now check whether the right hand side is a solution of this equation by simply substituting it for f in this differential equation and checking whether the resulting expression on the left hand side simplifies to zero. It does, but let us, for the purpose of illustration, pretend that the expression on the right hand side was not a simple explicit closed form. In this case, we would use closure properties to derive a differential equation for the right hand side. In our case, the right hand side is an algebraic function, which we may regard as an algebraic function substituted into the identity function (which is D-finite), so we can compute a differential equation for it. One of several possible results is

$$(1 - 2xz + z^2)g'(z) + (z - x)g(z) = 0.$$

Next, define $h = f - g$ and use a closure properties algorithm to compute a differential equation for h from the known equations for f and g. One of several possible results is

$$(z^3 - 2xz^2 + z)h^{(3)}(z) + (7z^2 - 9xz + 2)h''(z) + (10z - 6x)h'(z) + 2h(z) = 0.$$

(Incidentally this happens to be the same equation that we got for f, but in general we may get a larger equation at this step.) This differential equation has a three-dimensional vector space of solutions, and each particular solution h is uniquely determined by the values $h(0)$, $h'(0)$ and $h''(0)$. Since

$$h(z) := \sum_{n=0}^{\infty} P_n(x) z^n - \frac{1}{\sqrt{1 - 2xz + z^2}}$$

is a solution by construction, we have $h = 0$ if and only if $h(0) = h'(0) = h''(0) = 0$. In other words, the equation we constructed for the difference between left hand side and right hand side of the conjectured identity implies that the identity is true if and only if it is true for a certain (small) finite number of initial values. It is easy to check (with a computer) that the required terms match, and this completes the proof.

Further examples of this kind can be found in [89, 69, 56, 55, 51, 53].

2.4. **Evaluation.** Using the recurrence equation of a D-finite sequence, it is easy to compute many terms of the sequence recursively. Writing the linear recurrence

$$p_0(n)a_n + \cdots + p_r(n)a_{n+r} = 0$$

as a first order matrix equation,

$$\begin{pmatrix} a_{n+1} \\ a_{n+2} \\ \vdots \\ a_{n+r} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -\frac{p_0(n)}{p_r(n)} & -\frac{p_1(n)}{p_r(n)} & \cdots & \cdots & -\frac{p_{r-1}(n)}{p_r(n)} \end{pmatrix} \begin{pmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+r-1} \end{pmatrix},$$

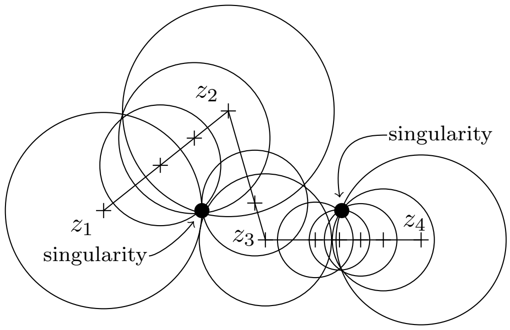
we see that the N-th term a_N, where N may be large, can be obtained from the terms $a_0, \ldots, a_{r-1}$ by roughly N matrix-vector multiplications. If we want to know all the terms $a_0, \ldots, a_N$, then the best way is to calculate all these multiplications and collect the results along the way. But if we only need the N-th term, there are better ways [18, 47].

In the differential case, when the operator algebra $A[\partial]$ acts on the space of meromorphic functions, we can uniquely specify a D-finite meromorphic function f by an operator L which annihilates it together with a suitable finite number of initial terms $f(0), f'(0), \ldots, f^{(r)}(0)$. To make things really feasible, we must assume that these initial values as well as the constants appearing in L belong to some computable subfield of $\mathbb{C}$. But even in this case, the value $f(z)$ of f at some point point z different from 0 will typically still be some transcendental number which does not admit any reasonable closed form. In view of hard open number theoretic questions which can be phrased as the evaluation of such values in "closed form" (e.g., $\zeta(5) \in \mathbb{Q}$?, $\pi+\exp(1) \in \mathbb{Q}$?), we should not expect an algorithm that produces some closed form representations of values of D-finite functions at some point z.

However, it is possible to compute the value $f(z)$ to arbitrary precision. This means that there is an algorithm which takes the following data as input:

- An operator $L = \ell_0 + \cdots + \ell_r\partial \in \mathbb{Q}[x][\partial]$ with $x \nmid \ell_r$ which annihilates the function f to be evaluated.
- some initial values $f(0), \ldots, f^{(r)}(0) \in \mathbb{Q} + \mathrm{i}\mathbb{Q}$
- a finite sequence of points $[z_1, \ldots, z_m] \in (\mathbb{Q} + \mathrm{i}\mathbb{Q})^m$ such that $z_1 = 0$ and no line segment $\overline{z_i z_{i+1}} \subseteq \mathbb{C}$ passes through a root of ℓ_r
- some number $N \in \mathbb{N}$

and which produces as output an approximation $u \in \mathbb{Q}+\mathrm{i}\mathbb{Q}$ with $|f(z_m)-u| < 2^{-N}$, where $f(z_m)$ is the value of f at the point z_m according to the analytic continuation along the path $z_1-z_2-\cdots-z_m$. Because of the assumption $x \nmid \ell_r$, the function f has no singularity at the origin, and its series expansion there converges. This expansion can be used to get arbitrary precision evaluations within the radius of convergence. In order to evaluate f at a point outside the radius of convergence, we choose a point on the given path $z_1-z_2-\cdots-z_m$ which is still inside the radius of convergence, not too close to its boundary and not too close to its center, and we determine approximations of the series coefficients of the expansion of f at that point. This new series will have a domain of convergence that goes at least a bit beyond the original disk of convergence. By repeating the procedure, we can work our way along the path until we obtain a series whose disk of convergence contains z_m. The process is illustrated in the figure below. The technical difficulties consist in making good choices for the truncation orders of the series, in ensuring that the accuracy of the approximations of the intermediate series expansions are high enough to guarantee the desired accuracy for the final output, and in keeping the computational cost low. See [29, 77, 78] for a detailed discussion of these issues. When $x \mid \ell_r$, the problem becomes much more difficult, but there are still algorithms available for this case [79].



Points in $\mathbb{C}$ where the leading coefficient $\ell_r \in \mathbb{C}[x]$ of a differential operator $L \in \mathbb{C}[x][\partial]$ vanishes are called singularities of the operator. In the specification above, we have assumed for simplicity that the input operator has no singularity at the point $z_1 = 0$, for which the initial values are supplied. In this case, it is guaranteed that the operator admits r linearly independent formal power series solutions. If there is a singularity at 0, it may still happen that there are r linearly independent formal power series solutions. In this case the singularity at 0 is called apparent. In general, however, the vector space generated by the formal power series solutions of L will have a dimension smaller than r when L has a singularity at zero. This reflects the fact that the operator has some solutions that do not admit a power series expansion at the origin. For example, the operator $x\partial + 1$ has the solution $\frac{1}{x}$, but no power series solution.

We can always get r linearly independent series solutions, even in the singular case, if we accept a sufficiently general notion of series. More specifically, for every differential operator $L \in \mathbb{C}[x][\partial]$ of order r there always exist r linearly independent series solutions of the form

$$\exp\bigl(p(x^{-1/s})\bigr)x^{\nu}a(x^{1/s}, \log(x)),$$

where $s \in \mathbb{N} \setminus \{0\}$, $p \in \mathbb{C}[x]$, $\nu \in \mathbb{C}$, and $a \in \mathbb{C}[[x]][y]$. For a given operator $L \in C[x][\partial]$, where C is some computable subfield of $\mathbb{C}$, it is possible to compute s, p, ν and any finite number of terms of the series a. The procedure is described in [44, 81, 55]. In the first place, generalized series solutions only have a formal algebraic meaning. But it is possible to associate to each such series solution an actual analytic function, defined on an appropriate subset of $\mathbb{C}$, which is a solution of the operator and whose asymptotic behaviour for $x \to 0$ is described by the generalized series. The correspondance is explained in [8].

There is also a notion of generalized series solutions for recurrence operators. In this case, they have the form

$$\Gamma(x)^{\gamma}\phi^{x}\exp(p(x^{1/s}))x^{\alpha}a(x^{-1/s}, \log(x))$$

for $\gamma \in \mathbb{Q}$, $\phi \in \mathbb{C}$, $s \in \mathbb{N}$, $p \in \mathbb{C}[x]$, $\alpha \in \mathbb{C}$, $a \in \mathbb{C}[[x]][y]$. Every recurrence operator $L \in \mathbb{C}[x][\partial]$ of order r admits r linearly independent solutions of this kind. For operators in $C[x][\partial]$ where C is a computable subfield of $\mathbb{C}$, truncations of these solutions at any desired order can be computed by a procedure which is similar to the procedure in the differential case. See [87, 50] for details and examples.

Generalized series solutions of recurrence operators can be viewed as asymptotic expansions for $x \to \infty$ of sequence solutions. This is similar to the differential case, although the theoretical justification of the correspondance is not yet as developped. In practice however, the accuracy of the correspondance is striking. Consider, as an example, the sequence $f\colon \mathbb{N} \to \mathbb{Z}$ defined by $f(0) = 1$, $f(1) = 4$, $f(2) = 24$ and

$$\begin{aligned}
5(n+1)(n+2)(74n+201)f(n)& \\
-16(n+2)(37n^2+156n+147)f(n+1)& \\
-8(296n^3+1988n^2+4376n+3147)f(n+2)& \\
+2(n+3)(2n+5)(74n+127)f(n+3) &= 0
\end{aligned}$$

for $n \geq 0$. The recurrence has the three generalized series solutions

$$\begin{aligned}
\frac{(\alpha/2)^n}{\sqrt{n}}\Bigl(1 &+ \tfrac{2128\alpha^2-29264\alpha-125849}{405224}n^{-1} + \tfrac{-1361376\alpha^2+20862240\alpha+27403225}{239892608}n^{-2} \\
&+ \tfrac{5(2345140112\alpha^2-37235246032\alpha-19495870819)}{2627303842816}n^{-3} \\
&- \tfrac{21(424579905600\alpha^2-6736276986560\alpha-4323871190307)}{3110727749894144}n^{-4} \\
&+ \tfrac{21(5238631753932208\alpha^2-80267077173522224\alpha-93005324288356259)}{34068690316840665088}n^{-5} \\
&+ \mathrm{O}(n^{-6})\Bigr)
\end{aligned}$$

where α may be any of the three roots of the polynomial $10 - 8x - 16x^2 + x^3$. This suggests that the asymptotic behaviour of f for $n \to \infty$ is given by $f(n) \sim \kappa\frac{(\alpha/2)^n}{\sqrt{n}}$, for some nonzero constant κ. The numerical values

$$\begin{aligned}
f(11)/f(10) &\approx 7.847370412 \\
f(101)/f(100) &\approx 8.183936697 \\
f(1001)/f(1000) &\approx 8.220582914
\end{aligned}$$

give an idea which of the three roots α is the correct one. (Indeed, the polynomial has a real root $\alpha = 16.44938308195571213\ldots$.)

In general, it is not easy to say something specific about the multiplicative constant κ, but at least we can use the series solutions to get a good numerica approximation of it. For the sake of illustration, let $E_0(n) = \frac{(\alpha/2)^n}{\sqrt{n}}$, $E_5(n)$ the truncated series quoted above (with the term $\mathrm{O}(n^{-6})$ replaced by 0), and $E_{10}(n)$ be the series solution truncated at order 10. Then we have $f(n)/E_i(n) \to \kappa$ for $n \to \infty$, but with different speed. The following table gives some idea:

	$\frac{f(n)}{E_0(n)}$	$\frac{f(n)}{E_5(n)}$	$\frac{f(n)}{E_{10}(n)}$
$n = 100$	$\underline{0.5}198357827$	$\underline{0.520238742137}60325251$	$\underline{0.520238742137467155914}832616826$
$n = 200$	$\underline{0.520}0371417$	$\underline{0.52023874213747}135185$	$\underline{0.520238742137467155914051}803075$
$n = 300$	$\underline{0.520}1043151$	$\underline{0.520238742137467}70599$	$\underline{0.5202387421374671559140510}64778$
$n = 400$	$\underline{0.520}1379118$	$\underline{0.520238742137467}28616$	$\underline{0.52023874213746715591405105}2546$
$n = 500$	$\underline{0.520}1580731$	$\underline{0.5202387421374671}9854$	$\underline{0.5202387421374671559140510518}96$

With $n = 10000$ we find

$$\frac{f(n)}{E_{10}(n)} \approx \underline{0.5202387421374671559140510518178442645572}3666550663586360,$$

and this may be enough accuracy to guess the closed form

$$\kappa \stackrel{?}{=} \sqrt{\beta/\pi} \approx \underline{0.52023874213746715591405105181784426455722908885022355854}$$

where β is the root of $2368x^3 - 1776x^2 - 196x - 5$ with value $\beta \approx 0.850267$.

2.5. **Closed Forms.** There are various algorithms for deciding whether a given D-finite function admits a certain type of closed form representation. The most simple question of this sort is whether a given D-finite recurrence or differential equation admits a polynomial solution.

It is easy to find all the polynomial solutions up to some prescribed degree. For example, to find all the cubic polynomials which are killed by a given operator $L \in A[\partial]$, just make an ansatz $c_0 + c_1x + c_2x^2 + c_3x^3$ with undetermined coefficients c_0, c_1, c_2, c_3, and apply L to this template. The unknown constants c_i are unknown, but they are known to be constants, so they commute with ∂. We can therefore write $L \cdot (c_0 + c_1x + c_2x^2 + c_3x^3)$ as a polynomial (or perhaps as a rational function) in x whose coefficients (in the numerator) are certain explicit linear combinations of the unknowns c_i. Equating all these linear forms to zero gives a homogeneous linear system the solutions of which are precisely the coefficient vectors of the cubic polynomial solutions of L.

Example: In the differential case, consider $L = (4x^2 - 9)\partial^2 - (4x^2 + 12x - 15)\partial + (12x - 6)$. Then

$$\begin{aligned}
&L \cdot (c_0 + c_1x + c_2x^2 + c_3x^3) \\
&= (2c_2 + 6c_3x)(4x^2 - 9) - (4x^2 + 12x - 15)(c_1 + 2c_2x + 3c_3x^2) \\
&\qquad + (12x - 6)(c_0 + c_1x + c_2x^2 + c_3x^3) \\
&= (-6c_0 + 15c_1 - 18c_2) + (12c_0 - 18c_1 + 30c_2 - 54c_3)x \\
&\qquad + (8c_1 - 22c_2 + 45c_3)x^2 + (4c_2 - 18c_3)x^3.
\end{aligned}$$

This is zero if and only if

$$\begin{pmatrix} -6 & 15 & -18 & 0 \\ 12 & -18 & 30 & -54 \\ 0 & 8 & -22 & 45 \\ 0 & 0 & 4 & -18 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = 0.$$

The solution space is generated by the vector $(27, 54, 36, 8)$. Therefore, the solutions of L of degree at most three are precisely the constant multiples of the polynomial $27 + 54x + 36x^2 + 8x^3$.

As the solution space of every operator is a finite dimensional vector space, it is clear that no operator can have polynomial solutions of unlimited degree. Instead, for every operator L there must be some finite $d \in \mathbb{N}$ such that all polynomial solutions of L must have degree d or less. If we can find some upper bound for d, then using this bound in the procedure sketched above will result in a basis of the vector space of all the polynomial solutions of a given operator. For the most common Ore algebras, it is not hard to determine such degree bounds for a given operator. See [68, 20, 56, 81, 70] for details of this construction. If the degree bounds turn out to be very high, efficient algorithms as those described in [16, 14] should be used instead of the naive algorithm.

Next, we might want to know whether a given operator has solutions that can be written as rational functions. The classical algorithm proceeds in two steps. First, it determines a polynomial d such that for every rational function solution f we have that df is a polynomial. In other words, d is (a multiple of) the least common denominator of all the rational function solutions. Such a polynomial d is called a denominator bound or a universal denominator for the operator. Algorithms for computing denominator bounds are available for various Ore algebras, see [81, 1, 83, 19] for details. Once a denominator bound is known, we can make an ansatz $f = p/d$ with an unknown polynomial p. Plugging this ansatz into the equation for f yields a new equation for p. The polynomial solutions p of that new equation are in one-to-one correspondance with the rational solutions $f = p/d$ of the original equation.

The next more general class of "closed forms" are those functions which have an annihilating operator of order 1. In the differential case, such function are called hyperexponential. In the shift-case, they are called hypergeometric terms. To decide whether an operator L admits a solution which is also a solution of some first-order operator is equivalent to the problem of deciding whether L can be written in the form $L = PQ$ for some operators P, Q where Q has order 1: it is clear that every solution of Q is also a solution of L, because $Q \cdot f = 0 \Rightarrow L \cdot f = PQ \cdot f = P \cdot (Q \cdot f) = P \cdot 0 = 0$. On the other hand, if f is a nonzero solution of both L and Q, then it is a solution of $\operatorname{gcrd}(L, Q)$, and since Q has order 1 and the only order 0 operator is 1, which does not have any nonzero solutions, we must have $\operatorname{gcrd}(L, Q) = Q$ and so in particular Q is a right divisor of L. For algorithms to find first-order right factors, see [67, 68, 82, 56, 84, 81, 34, 48].

The next more general class of closed form expressions are called d'Alembertian solutions. In the differential case, these are functions which can be written as iterated indefinite integrals over hyperexponential functions. An example is

$$f(x) = \int_0^x \exp(u) \int_0^u \frac{1}{v^2+1} \int_0^v \frac{\exp(w^2+w)}{w^4+5} dw\, dv\, du.$$

Analogously, in the shift case, we have functions that can be written as iterated indefinite sums over hypergeometric terms, for example

$$f(n) = \sum_{k=1}^{n} \binom{2k}{k} \sum_{i=1}^{k} \frac{2^i}{i^2+1} \sum_{j=1}^{i} \frac{1}{\binom{2j}{j+2}}.$$

In the general case, an element f of a function space F is called d'Alembertian if it is annihilated by some operator $L \in A[\partial]$ which can be written as a product of first-order operators. In order to determine the d'Alembertian solutions of a given operator L, first determine all the first-order right hand factors. Then for each such factor P, compute the first-order right hand factors of the operator obtained from L after dividing P from the right. Apply the algorithm recursively to this operator.

There are even more general algorithms for finding even more general types of closed form solutions. Most of them are however limited to particular Ore algebras. The best studied case is the differential case. Here we have algorithms for finding algebraic function solutions [72], liouvillean solutions [73], or even (certain) hypergeometric series solutions [36, 85, 59, 43]. For analogous results about recurrence operators, see for example [80, 42, 71, 24] and the references given there.

3. Several Variables

A function in several variables is called D-finite if it is D-finite with respect to each of the variables when all the other variables are considered as constant parameters.

3.1. **Examples.**

(1) $f(x, y) = x + \exp(xy)$ is D-finite because it satisfies the differential equation

$$(1 - xy)\frac{\partial^2}{\partial x^2} f(x, y) + xy^2 \frac{\partial}{\partial x} f(x, y) - y^2 f(x, y) = 0$$

with respect to x, viewing y as formal parameter, as well as the differential equation

$$\frac{\partial^2}{\partial y^2} f(x,y) - x\frac{\partial}{\partial y} f(x,y) = 0$$

with respect to y, viewing x as formal parameter.

(2) $f(n,k) = \binom{n}{k}$ is D-finite because it satisfies the recurrence

$$(n+1)f(n,k) - (n+1-k)f(n+1,k) = 0$$

with respect to n, viewing k as formal parameter, as well as the recurrence

$$(k-n)f(n,k) + (k+1)f(n,k+1) = 0$$

with respect to k, viewing n as formal parameter.

(3) $f(n,x) = x^n$ is D-finite because it satisfies the recurrence

$$f(n+1,x) - xf(n,x) = 0$$

with respect to n, viewing x as formal parameter, as well as the differential equation

$$x\frac{\partial}{\partial x} f(n,x) - nf(n,x) = 0$$

with respect to x, viewing n as formal parameter.

(4) The sequence $f(n,x) = P_n(x)$ of Legendre Polynomials is D-finite because it satisfies the recurrence

$$(1+n)f(n,x) - (3+2n)x\, f(n+1,x) + (2+n)f(n+2,x) = 0$$

with respect to n, viewing x as formal parameter, as well as the differential equation

$$n(n+1)f(n,x) - 2x\frac{\partial}{\partial x} f(n,x) + (1-x^2)\frac{\partial^2}{\partial x^2} f(n,x) = 0$$

with respect to x, viewing n as formal parameter.

(5) $f(n,x) = \sqrt{n+x}$ is not D-finite, although it satisfies the linear differential equation

$$2(n+x)\frac{\partial}{\partial x} f(n,x) - f(n,x) = 0$$

with respect to x, viewing n as formal parameter. The reason is that it does not satisfy any linear recurrence with respect to n, viewing x as formal parameter. (The absence of such a recurrence is not exactly obvious, but can be proven [40].)

Likewise, $f(n,x) = \Gamma(n,x)$ is not D-finite, although it satisfies a recurrence with respect to n, viewing x as formal parameter. The reason is that it does not satisfy any linear differential equation with respect to x, viewing n as a formal parameter.

3.2. **Algebraic Setup.** Also in the case of several variables, we can give a general definition of D-finiteness by using operators. The definition of Ore algebras quoted in Section 2.2 generalizes as follows.

Definition 3. *Let A be an integral domain, let $\sigma_1, \dots, \sigma_m \colon A \to A$ be endomorphisms with $\sigma_i \circ \sigma_j = \sigma_j \circ \sigma_i$ for all i, j, and let $\delta_i \colon A \to A$ be a σ_i-derivation for each $i = 1, \dots, m$.*

Let $A[\partial_1,\dots,\partial_m]$ be the set of all multivariate polynomials in the indeterminates $\partial_1,\dots,\partial_m$ with coefficients in A. Addition in $A[\partial_1,\dots,\partial_m]$ is defined coefficient-wise (as usual), and multiplication is defined via the commutation rules

$$\begin{aligned}\partial_i\partial_j &= \partial_j\partial_i && (i,j=1,\dots,m)\\ \partial_i a &= \sigma_i(a)\partial_i+\delta_i(a) && (i=1,\dots,m; a\in A).\end{aligned}$$

Then $\mathbb{A} := A[\partial_1,\dots,\partial_m] := (A[\partial_1,\dots,\partial_m],(\sigma_1,\dots,\sigma_m),(\delta_1,\dots,\delta_m))$ is called an Ore algebra over the ground domain A.

The most common examples are when A is a polynomial ring or a rational function field in m variables $x_1,\dots,x_m$, and each ∂_i is an operator which interacts with x_i but leaves the other variables fixed. For example, the mth Weyl algebra $\mathbb{C}[x_1,\dots,x_m][\partial_1,\dots,\partial_m]$ with $\sigma_i=\mathrm{id}$ and $\delta_i=\frac{d}{dx_i}$ for all i is an Ore algebra. Of course, with other choices of σ_i and δ_i, the indeterminates ∂_i can also be used to model shift or q-shift or other operators. It is not necessary that all the ∂_i are of the same type.

We use elements of Ore algebras to describe functions which live in an $A[\partial_1,\dots,\partial_m]$-left module F. Typically, the elements of F are functions in m variables $x_1,\dots,x_m$ and the operator ∂_i acts nontrivially on x_i and trivially on the other variables. For example, the Ore algebra $\mathbb{C}[x,y][\partial_x,\partial_y]$ of differential operators acts in a natural way on the space $\mathbb{C}[[x,y]]$ of bivariate formal power series.

Like in the univariate case, the *annihilator* of a fixed element $f\in F$ is defined as

$$\operatorname{ann}(f) := \{\, L\in A[\partial_1,\dots,\partial_m] : L\cdot f=0\,\}.$$

Also like in the univariate case, this is a left ideal of the Ore algebra, but unlike in the univariate case, it is usually not generated by a single element. The general definition of D-finiteness is as follows.

Definition 4. *Let $A[\partial_1,\dots,\partial_m]$ be an Ore algebra which acts on F. An element $f\in F$ is called D-finite (with respect to the action of $A[\partial_1,\dots,\partial_m]$ on F) if* $\operatorname{ann}(f)\cap A[\partial_i]\neq\{0\}$ *for all $i=1,\dots,m$.*

For $m=1$ this definition falls back to the definition given earlier for the univariate case. For arbitrary m, note that the definition says that the function f should be D-finite with respect to the restricted action of $A[\partial_i]$ on F, for every i. In other words, it should be D-finite with respect to every variable when all the other variables are viewed as formal parameters.

When A is a field, f is D-finite if and only if the left module $A[\partial_1,\dots,\partial_m]/\operatorname{ann}(f)$ is an A-vector space of finite dimension. In the univariate case, when $\operatorname{ann}(f)=\langle L\rangle$ for some operator L order r, the vector space is generated by $\{1,\partial,\dots,\partial^{r-1}\}$ and its dimension is r. In the multivariate case, when $\operatorname{ann}(f)\cap A[\partial_i]=\langle L_i\rangle$ for some operators L_i of order r_i ($i=1,\dots,m$), the vector space is generated by all the power products $\partial_1^{e_1}\cdots\partial_m^{e_m}$ with $0\le e_i\le r_i$ for $i=1,\dots,m$. These are $r_1r_2\cdots r_m<\infty$ many. However, this is in general only an upper bound for the dimension, because $\operatorname{ann}(f)$ may also contain mixed operators that cause the dimension to drop.

For example, for the Legendre polynomials $f(n,x)=P_n(x)$, we have

$$\begin{aligned}\operatorname{ann}(f)\cap\mathbb{Q}(n,x)[\partial_x] &= \langle\underbrace{n(n+1)-2x\partial_x+(1-x^2)\partial_x^2}_{=:L_x}\rangle,\\ \operatorname{ann}(f)\cap\mathbb{Q}(n,x)[\partial_n] &= \langle\underbrace{(1+n)-(2n+3)x\partial_n+(n+2)\partial_n^2}_{=:L_n}\rangle\end{aligned}$$

where ∂_x acts like the partial derivation in x and ∂_n acts as shift in n. In fact we even have $\operatorname{ann}(f)=\langle L_x,L_n\rangle$ in this case. Nevertheless, the dimension of the vector space $\mathbb{Q}(n,x)[\partial_x,\partial_n]/\operatorname{ann}(f)$ is strictly less than four. To see this, note that we have

$$\begin{aligned}&(n+1)x-(1-x^2)\partial_x-(n+1)\partial_n\\&=\Bigl(\tfrac{x}{2}-\tfrac{4+4n+n^2+15x^2+16nx^2+4n^2x^2}{2(n+1)(2n+5)}\partial_n+\tfrac{(n+2)x}{n+1}\partial_n^2-\tfrac{(n+2)(n+3)}{2(n+1)(2n+5)}\partial_n^3\Bigr)L_x\\&\quad+\Bigl(\tfrac{(n+2)x}{2}-\tfrac{1}{n+1}\partial_x-\tfrac{x(x^2-1)}{2(n+1)}\partial_x^2-\tfrac{(n+2)(12+7n+n^2)}{2(n+1)(2n+5)}\partial_n\\&\qquad\qquad+\tfrac{(n+2)x}{(n+1)(2n+5)}\partial_x\partial_n+\tfrac{(n+2)(x^2-1)}{2(n+1)(2n+5)}\partial_x^2\partial_n\Bigr)L_n\quad\in\operatorname{ann}(f).\end{aligned}$$

Using this relation, all operators involving some power of ∂_x can be rewritten in terms of some operator that only involves ∂_n, and using the generator of $\operatorname{ann}(f)\cap\mathbb{Q}(n,x)[\partial_n]$ we can rewrite any such operator as a $\mathbb{Q}(n,x)$-linear combination of 1 and ∂_n. Hence the dimension of $\mathbb{Q}(n,x)[\partial_x,\partial_n]$ is at most two.

3.3. **Gröbner Bases.** D-finite functions in several variables are specified by a basis of the annihilating ideal $\operatorname{ann}(f)$ and an appropriate sample of initial values. In typical example situations, we do not know such a basis for sure, but all we have are some operators $L_1,\dots,L_k\in A[\partial_1,\dots,\partial_m]$ such that $I:=\langle L_1,\dots,L_k\rangle\subseteq\operatorname{ann}(f)$. As far as algorithms for D-finiteness are concerned, it is fair to work with I instead of $\operatorname{ann}(f)$ provided that I has the property that $I\cap A[\partial_i]\neq\{0\}$ for all i.

Each $\operatorname{ann}(f)\cap A[\partial_i]$ contains some nonzero operator L_i, and if we define $I=\langle L_1,\dots,L_k\rangle$, then I obviously has the property $I\subseteq\operatorname{ann}(f)$ and $I\cap A[\partial_i]\neq\{0\}$ for all i. However, such a basis is not necessarily very convenient to work with. In practice, we typically do not need to know the pure operators L_i explicitly, but only a guarantee for their existence. We can therefore take the freedom to work with bases of I that are better suited for computations.

Gröbner bases [21] are a canonical choice. Because of space limitations, we shall assume that the reader is familiar with the theory of Gröbner bases in the commutative case, as described for example in [35, 9, 22]. The key message is then that despite the noncommutativity of Ore algebras, theory and algorithms carry over almost literally to left ideals of Ore algebras. The reason is essentially that for every admissible ordering $<$ on the power products $\partial_1^{e_1}\cdots\partial_m^{e_m}$, we still have $\operatorname{lt}(\tau p)=\tau\operatorname{lt}(p)$ for every term $\tau\in[\partial_1,\dots,\partial_m]$ and all $p\in A[\partial_1,\dots,\partial_m]$. The noncommutativity induced by the various σ_i and δ_i only affects the lower order terms. For simplicity, let us restrict the attention to Ore algebras $A[\partial_1,\dots,\partial_m]$ where A is a field.

Like in the commutative case, there is a reduction process by which terms in a given operator are successively replaced by terms that are smaller in the chosen term ordering until no further reduction is possible. If $r\in A[\partial_1,\dots,\partial_m]$ can be obtained by such a reduction process from $p\in A[\partial_1,\dots,\partial_m]$ by reducing with elements of $G\subseteq A[\partial_1,\dots,\partial_m]$, we write $r=\operatorname{red}(p,G)$. Also similar to the commutative case, G is called a Gröbner basis if any of the following equivalent conditions is satisfied:

- $\operatorname{red}(p,G)=0$ for all p in the left ideal generated by G
- $\operatorname{red}(p,G)$ is uniquely determined for all operators p.
- The set of all leading terms of elements of the left ideal generated by G agrees with the set of leading terms of multiples of elements of G.
- If $T\subseteq[\partial_1,\dots,\partial_m]$ is the set of all power products τ with $\operatorname{red}(\tau,G)=\tau$, then the set $\{\tau+\langle G\rangle:\tau\in T\}$ forms an A-vector space basis of $A[\partial_1,\dots,\partial_m]/\langle G\rangle$.

Also Buchberger's algorithm extends to the Ore setting in a natural way.

As an example, consider the Ore algebra $\mathbb{Q}(n,x)[\partial_n,\partial_x]$ where ∂_n acts as shift in n and ∂_x as derivation in x. Then

$$\begin{aligned} G = \bigl\{&(n+1)\partial_n + (1-x^2)\partial_x - (n+1)x,\\ &(1-x^2)\partial_x^2 - 2x\partial_x + n(n+1)\bigr\}\end{aligned}$$

is a Gröbner basis with respect to the lexicographic term order with $\partial_n > \partial_x$. Denote the two elements by g_1, g_2, respectively. To see that the operator

$$p = (n+2)\partial_n^2 - (2n+3)x\partial_n + (1+n)$$

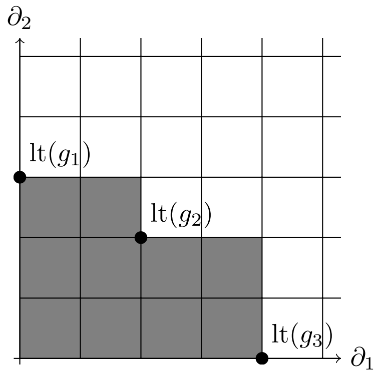
is contained in $\langle G\rangle$, observe that the reduction process

$$\begin{aligned} p \quad &\xrightarrow{-\partial_n g_1} \quad -(1-x^2)\partial_n\partial_x - (n+1)x\partial_n + (n+1)\\ &\xrightarrow{-\frac{-1+x^2}{n+1}\partial_x g_1} \quad -(n+1)x\partial_n + \frac{(x-1)^2(x+1)^2}{n+1}\partial_x^2\\ &\qquad\qquad + \frac{(n+3)(x-1)(x+1)x}{n+1}\partial_x + (n+x^2)\\ &\xrightarrow{+xg_1} \quad \frac{(x+1)^2(x-1)^2}{n+1}\partial_x^2 + \frac{2(x-1)(x+1)x}{n+1}\partial_x - n(x-1)(x+1)\\ &\xrightarrow{-\frac{x^2-1}{n+1}g_2} \quad 0\end{aligned}$$

leads to zero.

It follows from the theory of Gröbner bases that f is D-finite if and only if the Gröbner basis of $\operatorname{ann}(f)$ (with respect to any term order) contains for each $i = 1,\dots,m$ an element whose leading term is a power of ∂_i. While in theory this condition is equivalent to the condition that $\operatorname{ann}(f)\cap A[\partial_i] \neq \{0\}$ for all i, in practice it is less expensive to only ensure the pureness of the leading terms rather than of the entire operators. For example, for the Gröbner basis $\{g_1, g_2\}$ above, the operator g_1 is not pure, but since the leading terms ∂_n and ∂_x^2 are, the left ideal $\langle g_1, g_2\rangle$ corresponds to a D-finite function.

An ideal $I \subseteq A[\partial_1,\dots,\partial_m]$ contains for every $i = 1,\dots,m$ an element p whose leading term is a power of ∂_i if and only if this is the case for the Gröbner basis of I (with respect to any term order). The condition is also equivalent to saying that the set T of all power products τ with $\operatorname{red}(\tau, G)$ is finite, and therefore to saying that the A-vector space $A[\partial_1,\dots,\partial_m]/I$ has a finite dimension. This is again the finiteness which is referred to in the word D-finite.



The finiteness of $\dim_A A[\partial_1,\dots,\partial_m]/I$ makes it possible to treat multivariate D-finite functions in very much the same way as in the univariate case. In particular, we can execute closure properties of D-finite functions by solving certain linear systems.

Consider, as an example, the closure property plus. Suppose there are two D-finite functions f, g for which we know ideals $I, J \subseteq A[\partial_1, \dots, \partial_m]$ such that $\operatorname{ann}(f) \subseteq I$, $\operatorname{ann}(g) \subseteq J$, $\dim_A A[\partial_1, \dots, \partial_m]/I < \infty$, and $\dim_A A[\partial_1, \dots, \partial_m]/J < \infty$. We want to find a basis for the ideal $I \cap J$. All the operators in this ideal kill f as well as g, and hence every linear combination $\alpha f + \beta g$, and hence in particular the sum $f + g$.

As both I and J contain for every i some nonzero operator belonging to $A[\partial_i]$, it follows directly from the univariate argument that this is also the case for $I \cap J$. This argument also gives rise to an algorithm, but this algorithm is very inefficient. The method of choice avoids the computation of pure operators and works directly with the Gröbner bases of I and J, exploiting the fact that $p \in I \cap J \iff \operatorname{red}(p, \operatorname{Gb}(I)) = \operatorname{red}(p, \operatorname{Gb}(J)) = 0$. The idea behind the algorithm is the same as in the interpolation algorithm of Buchberger-Möller [23] and in the order-changing algorithm by Faugere-Gianni-Lazard-Mora known as FGLM [38]. The algorithm enumerates all the terms in increasing order and searches for linear relations among them. Whenever a relation is found, it is recorded, and all the multiples of its leading term are excluded from future consideration. After finitely many steps, no more terms are left, and then the recorded relations form a Gröbner basis of $I \cap J$.

INPUT: Gröbner bases $G, H \subseteq A[\partial_1, \dots, \partial_m]$ which contain for each i some nonzero operator whose leading term is a power of ∂_i.

OUTPUT: A Gröbner basis of the left ideal $\langle G\rangle \cap \langle H\rangle$.

1 $B = \emptyset$; $lt = \emptyset$; $done = \emptyset$

2 Let τ be the smallest power product which is not in *done* and which is not a multiple of some element of *lt*.

3 If no such τ exists, return B and stop.

4 Writing $done = \{\tau_1, \dots, \tau_k\}$, find, if possible, $\alpha_1, \dots, \alpha_k \in A$ such that

$$\operatorname{red}(\tau, G) = \alpha_1 \operatorname{red}(\tau_1, G) + \cdots + \alpha_k \operatorname{red}(\tau_k, G)$$
$$\operatorname{red}(\tau, H) = \alpha_1 \operatorname{red}(\tau_1, H) + \cdots + \alpha_k \operatorname{red}(\tau_k, H).$$

5 If such $\alpha_1, \dots, \alpha_k$ exist, then

6 set $B = B \cup \{\tau - \alpha_1\tau_1 - \cdots - \alpha_k\tau_k\}$ and $lt = lt \cup \{\tau\}$

7 otherwise

8 set $done = done \cup \{\tau\}$.

9 Go back to step 2.

The algorithms for the other closure properties are very similar. Only the condition in step 4 has to be adapted.

3.4. **Initial Values.** In general, given an ideal $I \subsetneq A[\partial_1, \dots, \partial_m]$ in an Ore algebra acting on a function space F, there are several elements of f whose annihilator contains I. In order to fix a specific element $f \in F$ annihilated by I, we need to supply some additional information to distinguish it for all the other functions annihilated by I.

The most easy case is the case of univariate recurrence equations whose leading coefficient is a polynomial with no positive integer roots,

$$p_0(n)a_n + p_1(n)a_{n+1} + \cdots + p_r(n)a_{n+r} = 0 \qquad (n \geq 0).$$

It is clear that any value a_n can be recursively computed via

$$a_n = -\frac{1}{p_r(n-r)}\bigl(p_0(n-r)a_{n-r} + \cdots p_1(n-r)a_{n-r+1} + \cdots + p_{r-1}(n-r)a_{n-1}\bigr)$$

if we specify any initial values $a_0, \dots, a_{r-1}$. The assumption that p_r has no positive integer roots ensures that there will be no division by zero. If p_r does have positive integer roots, say at $n_1, n_2, \dots, n_s \in \mathbb{N}$, then we have to specify the values a_{n_i+r} ($i = 1, \dots, s$) in addition to the initial values $a_0, \dots, a_{r-1}$ in order to fix the sequence.

The case of several variables is more subtle. Let us consider the situation in the shift case. Let $a_{n,k}$ be a D-finite sequence in n and k annihilated by some ideal $I = \operatorname{ann}(a_{n,k}) \subseteq \mathbb{Q}(n,k)[\partial_n, \partial_k]$. Suppose we know a Gröbner basis G of I. Without loss of generality, we may assume that the elements of I belong to $\mathbb{Q}[n,k][\partial_n, \partial_k]$. (If they don't, multiply them from the left by a suitable element of $\mathbb{Q}[n,k]$.)

If the leading coefficients $\operatorname{lc}(g) \in \mathbb{Q}[n,k]$ have no roots in $\mathbb{N}^2$, then the sequence $a_{n,k}$ is uniquely determined by I and the initial values $a_{i,j}$ for which (i,j) is such that $\operatorname{red}(\partial_n^i \partial_k^j, G) = \partial_n^i \partial_k^j$. Note that these are finitely many because $a_{n,k}$ is D-finite. To see that these values suffice, observe that for every other index (i', j') there exists a $g \in G$ whose leading term divides $\partial_n^{i'} \partial_k^{j'}$. By the assumption that the leading coefficient polynomials have no positive integer roots, we can use this g to express $a_{i',j'}$ as a finite linear combination of terms $a_{i'',j''}$ where (i'', j'') are smaller with respect to the term order. After repeating this procedure finitely many times, we are down to the initial values.

When the leading coefficients $\operatorname{lc}(g)$ do have roots in $\mathbb{N}^2$, we must be more careful. Unfortunately, this case appears frequently in typical examples. There are several points to be made:

- There is no algorithm for finding integer roots of multivariate polynomials, and so there is in general no way to find out whether there are any indices for which additional initial values have to be supplied. This is a severe theoretical restriction, but in practice it is typically not as dramatic as it seems.
- Polynomials in several variables may have infinitely many integer roots. This is bad if we want to describe a particular sequence by a finite amount of data.

  In practice, this situation is typically caused by integer-linear factors in the leading coefficients. This means that some subsequence of the form $a_{un,vn}$ for some specific $u, v \in \mathbb{N}$ remains undetermined by the recurrence system. But when $a_{n,k}$ is D-finite, we have reasons to hope that such a subsequence $a_{un,vn}$ is also D-finite, and we can specify the infinitely many terms of this sequence by a univariate recurrence and some finitely many initial values.
- For indices (i', j') to which the leading terms of several elements of G fit, it suffices if there is one of them whose leading coefficient does not vanish at this point. The critical points are thus only the common roots of the leading coefficients of all the elements of G whose leading coefficients fit.

  In general, the Gröbner basis of the annihilator of a D-finite sequence in d variables will have at least d elements, and all of them will fit to all the indices which are far enough from the boundary. It is fair to expect, although not guaranteed, that a system of d polynomials in d variables admits only finitely many solutions in $\mathbb{C}$. These can be computed, and the integer roots can be selected from them.

3.5. **Creative Telescoping.** In addition to the closure properties discussed in Section 3.3, which are direct generalizations of the closure properties known from the univariate case, there are also operations which are only meaningful in the case of several variables. For example, when $f(x, y)$ is a bivariate D-finite function, we can set one variable to a constant and regard the resulting object as a function with respect to the other variable. Is this function again D-finite? And if yes, how can we find an ideal of annihilating operators for it if we know an ideal of annihilating operators for the original function?

The key idea to approach these questions is the technique of creative telescoping. If we find operators $P, Q \in C[x, y][\partial_x, \partial_y]$ such that

$$(P + yQ) \cdot f = 0$$

then P is an annihilating operator for $f(x, 0)$. To see this, write the relation more verbosely,

$$\begin{aligned} &p_0(x)f(x, y) + p_1(x)\partial_x f(x, y) + \cdots + p_r(x)\partial_x^r f(x, y) \\ &+ y\Big(q_{0,0}(x, y)f(x, y) + \cdots + q_{u,v}(x, y)\partial_x^u\partial_y^v f(x, y)\Big) = 0, \end{aligned}$$

and observe that sending y to zero cancels the second line because of the leading factor y and leaves $P \cdot f(x, 0)$ in the first line because P is not affected by this substitution. (There is an implicit assumption that we do not encounter any singularity of f when y goes to zero for arbitrary but fixed x; let's ignore these details for the sake of simplicity.)

A relation $(P + yQ) \cdot f = 0$ is called a creative telescoping relation for f when P is nonzero. In this case, P is called a *telescoper* for f, and Q is called a *certificate* for f and P. Creative telescoping is the problem of finding P (and Q) for a given D-finite function f. More generally, instead of operators of the form $P+yQ$, it may also be of interest to search for operators of the form $P + \partial_y Q$, where ∂_y acts like the derivation in y. In this case (again assuming that f is sufficiently well-behaved) we have that

$$(P + \partial_y Q) \cdot f = 0$$

implies that $P \cdot F = h(x, 1) - h(x, 0)$ where $F(x) = \int_0^1 f(x, y)dy$ and $h = Q \cdot f$. Similarly, in the shift case, if f is annihilated by $P + (\partial_y - 1)Q$, then we can use P to derive an operator which annihilates the definite sum $F(n) := \sum_{k=0}^n f(n, k)$. This particular variant of the problem explains the name "creative telescoping": telescoping refers to the certificate part $(\partial_y - 1)Q$, which amounts to finding some anti-difference, and creative refers to the remaining degrees of freedom for choosing P in such a way that a suitable Q exists at all.

Examples:

(1) For the function $f(x, y) = \exp(\sqrt{1 + x^2 y} - x)$ we have

$$(1 + \partial_x) - y\,\frac{2}{x}\,\partial_y \in \operatorname{ann}(f).$$

It follows that $f(x, 0) = \exp(-x)$ is annihilated by $1 + \partial_x$.

(2) Consider the function $f(x, y) = \frac{1}{\sqrt{1-x^2+xy^3}}$ and consider the definite integral $F(x) = \int_0^1 f(x, y)dy$. Because of

$$\big((2x^2 - 1) + 3x(x^2 - 1)\partial_x\big) + \partial_y\,(1 + x^2)y \in \operatorname{ann}(f),$$

we have

$$\big((2x^2-1)+3x(x^2-1)\partial_x\big)\cdot F(x) = \int_0^1 \Big(\partial_y \cdot \frac{(1+x^2)y}{\sqrt{1-x^2+xy^3}}\Big)dy$$
$$= \Big[\frac{(1+x^2)y}{\sqrt{1-x^2+xy^3}}\Big]_{y=0}^1$$
$$= \frac{x^2+1}{\sqrt{1+x-x^2}}.$$

This is an inhomogeneous differential equation which together with the initial value $F(0)=1$ uniquely determines F.

(3) Let $f(n,k)=\binom{n}{k}\binom{n+k}{k}$ and $F(n)=\sum_{k=0}^n f(n,k)$. We have

$$\big((n+1)-3(2n+3)\partial_n+(n+2)\partial_n^2\big) + (\partial_k-1)\frac{2k^2(2n+3)}{(n-k+1)(n-k+2)} \in \operatorname{ann}(f).$$

Summing the equation

$$(n+1)f(n,k)-3(2n+3)f(n+1,k)+(n+2)f(n+2,k)$$
$$= (\partial_k-1)\cdot\frac{-2k^2(2n+3)}{(n-k+1)(n-k+2)}\binom{n}{k}\binom{n+k}{k}$$

for $k=0,\dots,n+2$ leads to

$$(n+1)F(n)-3(2n+3)F(n+1)+(n+2)F(n+2)$$
$$= \Big[\frac{-2k^2(2n+3)}{(n-k+1)(n-k+2)}\binom{n}{k}\binom{n+k}{k}\Big]_{k=0}^{n+3} = 0.$$

Note that the denominator $(n-k+1)(n-k+2)$ vanishes for the indices $k=n+1$ and $k=n+2$, which are in the summation range. This is common, but commonly not a problem, because the poles caused by the denominator are canceled by the roots of the binomials:

$$\frac{1}{(n-k+1)(n-k+2)}\binom{n}{k} = (n+1)(n+2)\binom{n+2}{k},$$

and the expression on the right is well-defined for all n and k.

As these examples show, it is easy to get from a creative telescoping relation for f to an annihilating operator for F. But does a creative telescoping relation always exist when f is D-finite? And can we find it algorithmically when we know an ideal basis of $\operatorname{ann}(f)$?

These questions have been intensively studied since the early 1990s, when Zeilberger proposed his holonomic systems approach [89] and presented his algorithm for definite hypergeometric summation [88, 90, 68]. This algorithm finds creative telescoping relations $P+(\partial_k-1)Q$ for the case when $f(n,k)$ is a hypergeometric term, whenever such a creative telescoping relation exists. A sufficient criterion for the existence is that the input hypergeometric term is "proper", see [68] for a definition. A necessary and sufficient condition was given by Abramov [2, 3].

An analogous algorithm for the differential case was given by Almkvist and Zeilberger [5]. It finds creative telescoping relations $P+\partial_y Q$ for hyperexponential functions. It can be shown that every hyperexponential function admits such a relation. An algorithm for the general case was given by Chyzak [30, 31, 32]. It finds a creative telescoping relation for an arbitrary D-finite function, provided there is one. This algorithm covers the shift case as well as the differential case, and it includes in particular the Zeilberger and the Almkvist-Zeilberger algorithm as special cases. A sufficient condition for the existence of a creative telescoping relation for a D-finite function f is that f is also "holonomic", see [30] for a definition.

These may be considered as the classical algorithms for creative telescoping. Earlier algorithmic approaches to the problem were based on elimination and Gröbner basis computations [37, 75, 76, 33]. Later algorithmic ideas include the algorithms of Apagodu and Zeilberger [6, 63, 7, 28], which are more efficient and easier to implement than the classical algorithms, and which also give easy access to sharp bounds on the sizes of telescopers in dependence of the input [27, 26].

The most recent line of development are reduction-based algorithms [12, 17, 13, 25]. These algorithms have the feature that they separete the computation of telescoper and certificate from each other. This is an interesting feature because the certificates are in general much larger than the telescopers, and in many applications they are not needed. Using these algorithms, it is possible to compute only a telescoper (without certificate) at a significantly lower cost than with the classical algorithms.

Another feature of reduction-based algorithms is that they are quite easy to explain. Let us give a sketch for the differential case when the integrand f is a bivariate rational function. Recall that by Hermite reduction [20, 86] we can write every rational function f in the form $f = \partial_y g + h$ where h is a rational function whose denominator is square free and whose numerator has a lower degree than its denominator. Every pole of h (viewed as a function in y) must already be a pole of f. Differentiation of f with respect to x does not introduce new poles but only affects their multiplicities. Therefore, if we apply Hermite reduction to the rational functions $f, \partial_x \cdot f, \cdots \partial_x^r \cdot f$, we obtain rational functions $g_0, \dots, g_r$ and $h_0, \dots, h_r$ with

$$\partial_x^i \cdot f = \partial_y \cdot g_i + h_i$$

for $i = 0, \dots, r$, and the denominators of all the h_i divide the square free part of the denominator of f. Writing d for this common denominator and $h_i = u_i/d$ for polynomials u_i in x and y ($i = 0, \dots, r$), find polynomials c_i in x only such that

$$c_0 u_0 + c_1 u_1 + \cdots + c_r u_r = 0.$$

This can be done by making an ansatz with undetermined coefficients, comparing coefficients with respect to y and solving a linear system. As the degrees of the u_i with respect to y are bounded by the degree of d in y, a nontrivial solution $(c_0, \dots, c_r)$ will exist as soon as r is sufficiently large. For every such solution $(c_0, \dots, c_r)$ we have

$$(c_0 + \cdots + c_r \partial_x^r) \cdot f = \partial_y \cdot (c_0 g_0 + \cdots + c_r g_r),$$

so $c_0 + \cdots + c_r \partial_x^r$ is a telescoper for f.

Creative telescoping is a very versatile technique. Several other interesting operations can be reduced to it. Here are some of them:

- If $f(x, y)$ is a bivariate Laurent series and $P - \partial_y Q$ annihilates f, then P annihilates the residue $\operatorname{res}_y f(x, y)$.
- More generally, if $f(x, y)$ is a bivariate Laurent series and $P - \partial_y Q$ annihilates $x^{1-n} f$, then P annihilates $[y^n] f(x, y)$.
- If $f(x, y) = \sum_{n,k=0}^{\infty} a_{n,k} x^n y^k$, we call $d(x) := \sum_{n=0}^{\infty} a_{n,n} x^n$ the diagonal of f. If $P - \partial_y Q$ annihilates $f(x, y/x)/y$, then P annihilates the diagonal d.
- Alternatively, if $P - k\,Q$ is a recurrence operator which annihilates the bivariate sequence $a_{n,k-n}$, then P annihilates its diagonal $a_{n,n}$.

4. Software

Implementations of most of the algorithms mentioned above are available for several computer algebra systems. For the univariate case, there are the Maple package gfun [69], a Mathematica package by Mallinger [61], and the Sage package ore_algebra [54]. For numerical evaluation of univariate D-finite functions, there is the numgfun package [62] which extends the functionality of gfun. For the multivariate case, there are the Maple package mgfun [31] and the Mathematica package HolonomicFunctions [57, 58]. There are also some special purpose systems which include functionality for efficient Gröbner basis computations in Operator algebra, for example the extension Plural [60] of Singular [41] or the system Risa/Asir [65].

References

[1] Sergei A. Abramov. Rational solutions of linear difference and q-difference equations with polynomial coefficients. In *Proceedings of ISSAC'95*, July 1995.

[2] Sergei A. Abramov. Applicability of Zeilberger's algorithm to hypergeometric terms. In *Proceedings of ISSAC'02*, pages 1–7, 2002.

[3] Sergei A. Abramov. When does Zeilberger's algorithm succeed? *Advances in Applied Mathematics*, 30(3):424–441, 2003.

[4] Milton Abramowitz and Irene A. Stegun. *Handbook of Mathematical Functions*. Dover Publications, Inc., 9th edition, 1972.

[5] Gert Almkvist and Doron Zeilberger. The method of differentiating under the integral sign. *Journal of Symbolic Computation*, 11(6):571–591, 1990.

[6] Moa Apagodu and Doron Zeilberger. Multi-variable Zeilberger and Almkvist-Zeilberger algorithms and the sharpening of Wilf-Zeilberger theory. *Advances in Applied Mathematics*, 37(2):139–152, 2006.

[7] Moa Apagodu and Doron Zeilberger. Multi-variable Zeilberger and Almkvist-Zeilberger algorithms and the sharpening of Wilf-Zeilberger theory. *Advances in Applied Mathematics*, 37(2):139–152, 2006.

[8] Werner Balser. *From Divergent Power Series to Analytic Functions*, volume 1582 of *Lecture Notes in Mathematics*. Springer-Verlag, 1994.

[9] Thomas Becker, Volker Weispfenning, and Heinz Kredel. *Gröbner Bases*. Springer, 1993.

[10] Bernhard Beckermann and George Labahn. A uniform approach for the fast computation of matrix-type Padé approximants. *SIAM Journal on Matrix Analysis and Applications*, 15(3):804–823, 1994.

[11] Alexandre Benoit, Frederic Chyzak, Alexis Darrasse, Stefan Gerhold, Marc Mezzarobba, and Bruno Salvy. The dynamic dictionary of mathematical functions. In *Proceedings of ICMS'10*, 2010. `http://ddmf.msr-inria.inria.fr/1.9.1/ddmf`.

[12] Alin Bostan, Shaoshi Chen, Frédéric Chyzak, and Ziming Li. Complexity of creative telescoping for bivariate rational functions. In *Proceedings of ISSAC'10*, pages 203–210, 2010.

[13] Alin Bostan, Shaoshi Chen, Frederic Chyzak, Ziming Li, and Guoce Xin. Hermite reduction and creative telescoping for hyperexponential functions. In *Proceedings of ISSAC'13*, pages 77–84, 2013.

[14] Alin Bostan, Frederic Chyzak, Thomas Cluzeau, and Bruno Salvy. Low complexity algorithms for linear recurrences. In Jean-Guillaume Dumas, editor, *Proceedings of ISSAC'06*, pages 31–39, 2006.

[15] Alin Bostan, Frederic Chyzak, Ziming Li, and Bruno Salvy. Fast computation of common left multiples of linear ordinary differential operators. In *Proceedings of ISSAC'12*, pages 99–106, 2012.

[16] Alin Bostan, Thomas Cluzeau, and Bruno Salvy. Fast algorithms for polynomial solutions of linear differential equations. In *Proceedings of ISSAC'05*, pages 45–52, 2005.

[17] Alin Bostan, Pierre Lairez, and Bruno Salvy. Creative telescoping for rational functions using the Griffith-Dwork method. In *Proceedings of ISSAC'13*, pages 93–100, 2013.

[18] Richard Brent and Paul Zimmermann. *Modern Computer Arithmetic*. Cambridge University Press, 2011.

[19] Manuel Bronstein. On solutions of linear ordinary difference equations in their coefficient field. *Journal of Symbolic Computation*, 29:841–877, 2000.

[20] Manuel Bronstein. *Symbolic Integration I*, volume 1 of *Algorithms and Computation in Mathematics*. Springer, 2nd edition, 2005.

[21] Bruno Buchberger. *Ein Algorithmus zum Auffinden der Basiselemente des Restklassenrings nach einem nulldimensionalen Polynomideal*. PhD thesis, Universität Innsbruck, 1965.

[22] Bruno Buchberger and Manuel Kauers. Gröbner basis. *Scholarpedia*, 5(10):7763, 2010. `http://www.scholarpedia.org/article/Groebner_basis`.

[23] Bruno Buchberger and Hans Michael Möller. The construction of multivariate polynomials with preassigned zeros. In *Proceedings of EUROCAM'82*, pages 24–31, 1982.

[24] Yongjae Cha. *Closed Form Solutions of Linear Difference Equations*. PhD thesis, Florida State University, 2010.

[25] Shaoshi Chen, Hui Huang, Manuel Kauers, and Ziming Li. A modified Abramov-Petkovsek reduction and creative telescoping for hypergeometric terms. In *Proceedings of ISSAC'15*, pages 117–124, 2015.

[26] Shaoshi Chen and Manuel Kauers. Order-degree curves for hypergeometric creative telescoping. In *Proceedings of ISSAC'12*, pages 122–129, 2012.

[27] Shaoshi Chen and Manuel Kauers. Trading order for degree in creative telescoping. *Journal of Symbolic Computation*, 47(8):968–995, 2012.

[28] Shaoshi Chen, Manuel Kauers, and Christoph Koutschan. A generalized apagodu-zeilberger algorithm. In *Proceedings of ISSAC'14*, pages 107–114, 2014.

[29] David V. Chudnovsky and Gregory V. Chudnovsky. Computer algebra in the service of mathematical physics and number theory. In David V. Chudnovsky and Richard D. Jenks, editors, *Computers in Mathematics*, volume 125 of *Lecture Notes in Pure and Applied Mathematics*, pages 109–232, Stanford University, 1986. Dekker.

[30] Frédéric Chyzak. Gröbner bases, symbolic summation and symbolic integration. In *Gröbner Bases and Applications*. Cambridge University Press, 1997.

[31] Frédéric Chyzak. *Fonctions holonomes en calcul formel*. PhD thesis, INRIA Rocquencourt, 1998.

[32] Frédéric Chyzak. An extension of Zeilberger's fast algorithm to general holonomic functions. *Discrete Mathematics*, 217:115–134, 2000.

[33] Frédéric Chyzak and Bruno Salvy. Non-commutative elimination in Ore algebras proves multivariate identities. *Journal of Symbolic Computation*, 26:187–227, 1998.

[34] Thomas Cluzeau and Mark van Hoeij. Computing hypergeometric solutions of linear recurrence equations. *Applicable Algebra in Engeneering, Communication and Computing*, 17:83–115, 2006.

[35] David Cox, John Little, and Donal O'Shea. *Ideals, Varieties, and Algorithms*. Springer, 1992.

[36] Ruben Debeerst, Mark van Hoeij, and Wolfram Koepf. Solving differential equations in terms of bessel functions. In *Proceedings of ISSAC'08*, pages 39–46, 2008.

[37] Sister Mary Celine Fasenmyer. A note on pure recurrence relations. *The American Mathematical Monthly*, 56:14–17, 1949.

[38] J.-Ch. Faugere, P. Gianni, D. Lazard, and T. Mora. Efficient computation of zero dimensional Gröbner bases by change of ordering. *Journal of Symbolic Computation*, 16(4):329–344, 1993.

[39] Philippe Flajolet, Stefan Gerhold, and Bruno Salvy. On the non-holonomic character of logarithms, powers, and the n-th prime function. *The Electronic Journal of Combinatorics*, 11(2):A2, 2005.

[40] Stefan Gerhold. On some non-holonomic sequences. *Electronic Journal of Combinatorics*, 11(1):1–8, 2004.

[41] Gert-Martin Greuel and Gerhald Pfister. *A Singular Introduction to Commutative Algebra*. Springer, 2002.

[42] P.A. Hendriks and M.F. Singer. Solving difference equations in finite terms. *Journal of Symbolic Computation*, 27(3):239–259, 1999.

[43] Erdal Imamoglu and Mark van Hoeij. Computing hypergeometric solutions of second order linear differential equations using quotients of formal solutions. In *Proceedings ISSAC'15*, pages 235–242, 2015.

[44] E. L. Ince. *Ordinary Differential Equations*. Dover, 1926.

[45] Maximilian Jaroschek. Improved polynomial remainder sequences for Ore polynomials. *Journal of Symbolic Computation*, 58:64–76, 2013.

[46] Maximilian Jaroschek. *Removable Singularities of Ore Operators*. PhD thesis, RISC, JKU, 2013.

[47] Fredrik Johansson. *Fast and rigorous computation of special functions to high precision*. PhD thesis, RISC, JKU, 2014.

[48] Fredrik Johansson, Manuel Kauers, and Marc Mezzarobba. Finding hyperexponential solutions of linear odes by numerical evaluation. In Manuel Kauers, editor, *Proceedings of ISSAC'13*, pages 211–218, 2013.

[49] Manuel Kauers. Guessing handbook. Technical Report 09-07, RISC-Linz, 2009.

[50] Manuel Kauers. A Mathematica package for computing asymptotic expansions of solutions of p-finite recurrence equations. Technical Report 11-04, RISC-Linz, 2011.

[51] Manuel Kauers. The holonomic toolkit. In Johannes Blümlein and Carsten Schneider, editors, *Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions*, Texts and Monographs in Symbolic Computation, pages 119–144. Springer, 2013.

[52] Manuel Kauers. Bounds for D-finite closure properties. In *Proceedings of ISSAC'14*, pages 288–295, 2014.

[53] Manuel Kauers. Computer algebra. In Miklos Bona, editor, *Handbook of Enumerative Combinatorics*, pages 975–1046. Taylor and Francis, 2015.

[54] Manuel Kauers, Maximilian Jaroschek, and Fredrik Johansson. Ore polynomials in Sage. In *Computer Algebra and Polynomials*, LNCS 8942, pages 105–125. Springer, 2014.

[55] Manuel Kauers and Peter Paule. *The Concrete Tetrahedron.* Springer, 2011.

[56] Wolfram Koepf. *Hypergeometric Summation.* Vieweg, 1998.

[57] Christoph Koutschan. *Advanced Applications of the Holonomic Systems Approach.* PhD thesis, RISC-Linz, Johannes Kepler Universität Linz, 2009.

[58] Christoph Koutschan. HolonomicFunctions (User's Guide). Technical Report 10-01, RISC Report Series, University of Linz, Austria, January 2010.

[59] Vijay Jung Kunwar and Mark van Hoeij. Second order differential equations with hypergeometric solutions of degree three. In *Proceedings of ISSAC'13*, pages 235–242, 2013.

[60] Viktor Levandovskyy and Hans Schönemann. Plural: A computer algebra system for noncommutative polynomial algebras. In *Proceedings of ISSAC'03*, pages 176–183, 2003.

[61] Christian Mallinger. Algorithmic manipulations and transformations of univariate holonomic functions and sequences. Master's thesis, J. Kepler University, Linz, August 1996.

[62] Marc Mezzarobba and Bruno Salvy. Effective Bounds for P-Recursive Sequences. *Journal of Symbolic Computation*, 45(10):1075–1096, 2010.

[63] Mohamud Mohammed and Doron Zeilberger. Sharp upper bounds for the orders of the recurrences outputted by the Zeilberger and q-Zeilberger algorithms. *Journal of Symbolic Computation*, 39(2):201–207, 2005.

[64] NIST. The digital library of mathematical functions. http://dlmf.nist.gov/.

[65] Masayuki Noro and Taku Takeshima. Risa/asir – a computer algebra system, 1992.

[66] O. Ore. Theory of non-commutative polynomials. *Annals of Mathematics*, 34:480–508, 1933.

[67] Marko Petkovšek. Hypergeometric solutions of linear recurrences with polynomial coefficients. *Journal of Symbolic Computation*, 14(2–3):243–264, 1992.

[68] Marko Petkovšek, Herbert Wilf, and Doron Zeilberger. $A = B$. AK Peters, Ltd., 1997.

[69] Bruno Salvy and Paul Zimmermann. Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable. *ACM Transactions on Mathematical Software*, 20(2):163–177, 1994.

[70] Carsten Schneider. Degree bounds to find polynomial solutions of parameterized linear difference equations in ΠΣ-fields. *Applicable Algebra in Engeneering, Communication and Computing*, 16(1):1–32, 2005.

[71] Carsten Schneider. Solving parameterized linear difference equations in terms of indefinite nested sums and products. *Journal of Difference Equations and Applications*, 11(9):799–821, 2005.

[72] Michael Singer. Algebraic relations among solutions of linear differential equations. *Transactions of the AMS*, 295(2):753–763, 1986.

[73] Michael F. Singer. Liouvillian solutions of linear differential equations with liouvillian coefficients. *Journal of Symbolic Computation*, 11(3):251–273, 1991.

[74] Richard P. Stanley. Differentiably finite power series. *European Journal of Combinatorics*, 1:175–188, 1980.

[75] Nobuki Takayama. An algorithm of constructing the integral of a module. In *Proceedings of ISSAC'90*, pages 206–211, 1990.

[76] Nobuki Takayama. Gröbner basis, integration and transcendental functions. In *Proceedings of ISSAC'90*, pages 152–156, 1990.

[77] Joris van der Hoeven. Fast evaluation of holonomic functions. *Theoretical Computer Science*, 210(1):199216, 1999.

[78] Joris van der Hoeven. Fast evaluation of holonomic functions near and in singularities. *Journal of Symbolic Computation*, 31(6):717–743, 2001.

[79] Joris van der Hoeven. Efficient accelero-summation of holonomic functions. *Journal of Symbolic Computation*, 42(4):389–428, 2007.

[80] Marius van der Put and Michael Singer. *Galois Theory of Difference Equations*, volume 1666 of *Lecture Notes in Mathematics*. Springer, 1997.

[81] Marius van der Put and Michael Singer. *Galois Theory of Linear Differential Equations.* Springer, 2003.

[82] Mark van Hoeij. Factorization of differential operators with rational functions coefficients. *Journal of Symbolic Computation*, 24:537–561, 1997.
[83] Mark van Hoeij. Rational solutions of linear difference equations. In *Proceedings of ISSAC'98*, pages 120–123, 1998.
[84] Mark van Hoeij. Finite singularities and hypergeometric solutions of linear recurrence equations. *Journal of Pure and Applied Algebra*, 139:109–131, 1999.
[85] Mark van Hoeij and Quan Yuan. Finding all bessel type solutions for linear differential equations with rational function coefficients. In *Proceedings of ISSAC'10*, pages 37–44, 2010.
[86] Joachim von zur Gathen and Jürgen Gerhard. *Modern Computer Algebra*. Cambridge University Press, 1999.
[87] Jet Wimp and Doron Zeilberger. Resurrecting the asymptotics of linear recurrences. *Journal of Mathematical Analysis and Applications*, 111:162–176, 1985.
[88] Doron Zeilberger. A fast algorithm for proving terminating hypergeometric identities. *Discrete Mathematics*, 80:207–211, 1990.
[89] Doron Zeilberger. A holonomic systems approach to special function identities. *Journal of Computational and Applied Mathematics*, 32:321–368, 1990.
[90] Doron Zeilberger. The method of creative telescoping. *Journal of Symbolic Computation*, 11:195–204, 1991.

Manuel Kauers, Institute for Algebra, J. Kepler University Linz, Austria
E-mail address: manuel.kauers@jku.at