Removing Apparent Singularities of Linear Differential Systems with Rational Function Coefficients

Moulay Barkatou

Université de Limoges ; CNRS - XLIM UMR 7252, FRANCE

Journées Nationales de Calcul Formel (JNCF) 2015
Cluny 2–6 Novembre 2015

Joint work with Suzy Maddah

Notation-Vocabulary

$\mathbf{K} = \mathbb{C}(z)$, $' = \partial = \frac{d}{dz}$.

System of first order linear differential equations:

$$[A] \qquad \partial X = A(z)X,$$

where $X = (x_1, \ldots, x_n)^T$ is column-vector of length n.

$A(z)$ is an $n \times n$ matrix with entries in $\mathbf{K} = \mathbb{C}(z)$.

The (finite) singularities of system $[A]$ are the poles of the entries of $A(z)$.

Scalar linear differential equation of order n: $L(x(z)) = 0$

$$L = \partial^n + c_{n-1}(z)\partial^{n-1} + \cdots + c_0(z) \in \mathbf{K}[\partial]$$

The (finite) singularities of L are the poles of the c_i's.

Apparent singularities

- Singularities of solutions of $L(x) = 0$ (resp. $[A]$) are necessarily singularities of the coefficients of L (resp. $[A]$), but the converse is not always true.

Def. An apparent singularity of L (resp. $[A]$) is a singular point where the general solution of $L(x) = 0$ (resp. $[A]$) is holomorphic.

Example 1. $L(x) = \frac{dx}{dz} - \frac{\mu}{z}x = 0, \qquad \mu \in \mathbb{C}.$

- The general solution of L is

$$x(z) = cz^{\mu}, \quad c \in \mathbb{C}.$$

- When $\mu \in \mathbb{N}$, the general solution of $L(x) = 0$ is holomorphic at $z = 0$.
- When $\mu \in \mathbb{N}$, the point $z = 0$ is an apparent singularity of L.

Task:
To detect and remove the apparent singularities of a given operator L or system $[A]$.

• Removing apparent singularities of $L \in \mathbb{C}(z)[\partial]$:

$\rightarrow$ to construct another operator $\tilde{L} \in \mathbb{C}(z)[\partial]$ such that:

(i) any solution of $L(y) = 0$ is a solution of $\tilde{L}(y) = 0$,
i.e. $\tilde{L} = R \circ L$ for some $R \in \mathbb{C}(z)[\partial]$

(ii) and the singularities of $\tilde{L}$ are exactly the singularities of L that are not apparent.

- Such an operator $\tilde{L}$ is called a desingularization of L.

Example: $L = \partial - \frac{\mu}{z}, \qquad \mu \in \mathbb{N}.$

The operator $\tilde{L} = \partial^{\mu+1}$ is a desingularization of L.

• Several algorithms have been developed for linear differential (and more generally Ore) operators, e.g.

- Abramov-Barkatou-van Hoeij'2006,
- M. Jaroschek '2013
- Chen-Jaroschek-Kauers-Singer'2013, Chen-Kauers-Singer'2015

- We developed, in [ABH 2006]* an algorithm that, given an operator L of order n, produces a desingularization $\tilde{L}$ with minimal order $m \geq n+1$.
- This algorithm has been implemented in Maple.
- I will refer to this algorithm as ABH method.

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* S. Abramov, M. Barkatou and M. van Hoeij AAECC 2006

Example 2:

Consider the second order operator

$$L := \partial^2 - \frac{(z+2)}{z}\partial + \frac{2}{z}.$$

- $z = 0$ is a singularity of L.
- The general solution of $L(y) = 0$ is given by

$$c_1 e^z + c_2 \left(1 + z + \frac{z^2}{2}\right) \qquad c_1, c_2 \in \mathbb{C}.$$

- L has an apparent singularity at $z = 0$.
- The desingularization computed by ABH method is of order 4

$$\tilde{L} = \partial^4 + \left(-1 + \frac{z}{4}\right)\partial^3 + \left(-\frac{1}{4} - \frac{3\,z}{8}\right)\partial^2 + \left(\frac{1}{2} + \frac{z}{8}\right)\partial - \frac{1}{4}$$

- The apparent singularity of L at $z = 0$ can be removed by computing a *gauge equivalent* first-order differential system with coefficient in $\mathbb{C}(z)$ of size $\mathrm{ord}(L) = 2$.
- Consider the first-order differential system associated with L

$$[A] \qquad \frac{d}{dz}X = A(z)X, \quad A(z) = \begin{bmatrix} 0 & 1 \\ \frac{-2}{z} & 1 + \frac{2}{z} \end{bmatrix}.$$

- Set

$$X = T(z)\ Y, \quad \text{where} \quad T(z) = \begin{bmatrix} 1 & 0 \\ 1 & z^2 \end{bmatrix}.$$

- The new variable Y satisfies the gauge equivalent first-order differential system of the same dimension given by

$$[B] \quad \frac{d}{dz}Y = B\ Y$$

where

$$B := T^{-1}AT - T^{-1}\frac{d}{dz}T = \begin{bmatrix} 1 & z^2 \\ 0 & 0 \end{bmatrix}.$$

- General fact : Any system $[A]$ with rational coefficients can be reduced to a *gauge equivalent* system $[B]$ with rational coefficients, such that the finite singularities of $[B]$ coincide with the non-apparent singularities of $[A]$.
- We present an algorithm which for a system $[A]$ constructs a desingularization $[B]$.

Outline:

1. Review of ABH method
2. New algorithm
3. Examples of comparison to ABH and existing methods for scalar equations
4. Conclusion

Desingularization of scalar equations
Review of ABH method

Classification of Singularities

Let $L \in \mathbb{C}(z)[\partial]$, $\partial = \frac{d}{dz}$, be monic, have order n :

$$L = \partial^n + c_{n-1}(z)\partial^{n-1} + \cdots + c_0(z).$$

- Let $S(L)$ be the set of finite singularities of L (poles of the c_i's.)
- $z_0 \in S(L)$ is a regular singularity if there exist $\underline{n \text{ linearly independent}}$ formal solutions at $z = z_0$ of the form

$$y_i = t^{\lambda_i} \left(\varphi_{i0}(t) + \varphi_{i1}(t) \log t + \cdots + \varphi_{is_i}(t)(\log t)^{s_i - 1}\right)$$

  where

  $$t = z - z_0, \quad 1 \leq s_i \leq n,\ \lambda_i \in \mathbb{C},\ \varphi_{ij} \in \mathbb{C}[[t]].$$

  The λ_i's called local exponents (at $z = z_0$)

- otherwise, z_0 is called an irregular singular point.

A point $z_0 \in \mathbb{C}$ is an ordinary point for L if $z_0 \notin S(L)$.

Proposition1 The following statements are equivalent.

a) z_0 is an ordinary point of L.

b) There exist a basis of solutions $y_1, \ldots, y_n$ of L, holomorphic at $z = z_0$, for which y_i vanishes at z_0 with order $i - 1$.

c) z_0 is not an irregular singularity of L, the local exponents at z_0 are $0, 1, \ldots, n-1$, and the formal solutions of L at z_0 are in $\mathbb{C}[[z - z_0]]$.

Characterization of apparent singularities

- Any apparent singularity is a regular singularity with n distinct integer exponents $0 \leq \lambda_1 < \cdots < \lambda_n$ and $\lambda_n \geq n$.

Proposition2 The following statements are equivalent.

a) z_0 is either an ordinary point or an apparent singularity of L.

b) z_0 is not an irregular singular point of L, the local exponents are non-negative integers and the formal solutions at z_0 do not involve logarithms.

c) There exists a monic operator $\tilde{L} \in \mathbb{C}(z)[\partial]$ that has L as a right-hand factor such that z_0 is an ordinary point of $\tilde{L}$.

Proof of Proposition 2:

- a) $\Rightarrow$ b) is clear.
- c) $\Rightarrow$ a) follows from Cauchy's theorem.
- b) $\Rightarrow$ c).
  - Let $m(z_0)$ be the highest local exponent at z_0.
  - Let $E(z_0) \subset \{0, 1, \ldots, m(z_0)\}$ be the set of exponents of L at z_0.
  - Let L_1 be an operator with the following as basis of solutions:

    $L((z - z_0)^i)$, $i \in \{0, 1, \ldots, m(z_0)\} \setminus E(z_0)$.
  - Then $\tilde{L} = L_1 L$ satisfies part b) of Proposition 1.
  - z_0 is an ordinary point of $\tilde{L}$.
  - Note that $\tilde{L}$ has order $m(z_0) + 1$.

Theorem Every monic $L \in \mathbb{C}(z)[\partial]$ has a desingularization $\tilde{L}$.

Proof:

- Let $A(L) := \{ \text{ finite apparent singularities of } L\}$, and

$$m := \max_{z_0 \in A(L)} m(z_0).$$

- If a desingularization $\tilde{L}$ exists then the order of $\tilde{L}$ must be at least $m+1$.

  Take $z_0 \in A(L)$ for which $m(z_0) = m$. m is an exponent of $\tilde{L}$ at z_0 (for L is a right-hand factor of $\tilde{L}$.),

  but since z_0 is a regular point of $\tilde{L}$ it follows that $0, 1, \ldots, m$ are exponents of $\tilde{L}$ at z_0 as well, so the order of $\tilde{L}$ must be at least $m+1$.

- We will now show that a desingularization $\tilde{L}$ of order $m+1$ exists.

How to construct a desingularization $\tilde{L}$ of order $m+1$?

- Construct $y_1, \ldots, y_{m+1-n} \in \mathbb{C}[z]$ such that for every $z_0 \in A(L)$ and every $i \in \{0, 1, \ldots, m\} \setminus E(z_0)$ there is precisely one y_j that vanishes at z_0 with order i.

- Let L_1 be the monic operator whose solutions are spanned by $L(y_1), \ldots, L(y_{m+1-n})$.

- Then every $z_0 \in A(L)$ is an ordinary point of $L_1 L$.

- However, $L_1 L$ need not satisfy the definition of a desingularization because we may have created new apparent singularities.

An example

Let L be the monic operator with $z\cos(z)$ and $z\sin(z)$ as solutions:

$$L = \partial^2 - \frac{2}{z}\partial + 1 + \frac{2}{z^2}$$

L has one apparent singularity at $z = 0$ with exponents 1 and 2.

- To desingularize L we must add a solution with exponent 0.
- Take $y_1 = z^0 = 1$ and compute $L(y_1)$. We find $L(y_1) = 1 + 2/z^2$.
- Let L_1 be the monic operator with $1 + 2/z^2$ as a basis of solutions :

$$L_1 = \partial + \frac{4}{z(z^2+2)}.$$

- Multiplying L on the left by L_1 adds a solution (namely $y_1 = z^0$) to L with the missing exponent 0.

$$L_1 L = \partial^3 - 2\,\frac{z\partial^2}{z^2+2} + \frac{\left(6+z^2\right)\partial}{z^2+2}$$

- Hence $z = 0$ is a regular point of $L_1 L$.
- Unfortunately, L_1 introduces new singularities, namely at $z^2 + 2 = 0$.

How to remedy this?

1. Let $a \in \mathbb{C}[z]$ be the denominator of $L_1 L$ so that $aL_1 L \in \mathbb{C}[z][\partial]$
2. Let $b \in \mathbb{C}[z]$ be the denominator of L so that $bL \in \mathbb{C}[z][\partial]$.
3. Let $d = gcd(a, b)$. Compute $u, v \in \mathbb{C}[z]$ such that $ua + vb = d$.
4. Now let $L' = uaL_1 L + v\partial^{m+1-n}.(bL)$
5. $L' \in \mathbb{C}[z][\partial]$ and its leading coefficient is d
6. The monic operator $\tilde{L} = \frac{1}{d} L'$ is a desingularization of L.

- We will illustrate how to remedy this using the trick above:

$$L = \partial^2 - \frac{2}{z}\partial + 1 + \frac{2}{z^2}$$

$$L_1 L = \partial^3 - 2\,\frac{z\partial^2}{z^2+2} + \frac{\left(6+z^2\right)\partial}{z^2+2}$$

- Here

$$a := z^2+2,\ \ b := z^2,$$

$$d := gcd(a,b) = 1, ua + vb = d,\ \ u = \frac{1}{2},\ v = -\frac{1}{2}$$

- Let $L' = uaL_1L + v\partial.(bL)$

$$\tilde{L} = L' = \partial^3 - z\partial^2 + 3\partial - z.$$

Example

- Consider
$$L=\partial^2+\frac{\left(3\,z^2-4\right)\partial}{z\,(z^2+2)}-2\,\frac{-1+2\,z^2}{z^2+2}$$
- L has an apparent singularity at $z=0$ with local exponents 0 and 3.

- The desingularization computed by ABH method is of order 4
$$\tilde{L}=\partial^4+1/2\,\frac{z\left(24+7\,z^2\right)\partial^3}{z^2+2}+1/2\,\frac{\left(58\,z^2+88+27\,z^4\right)\partial^2}{\left(z^2+2\right)^2}$$
$$-1/2\,\frac{z\left(-4\,z^2+4+93\,z^4+28\,z^6\right)\partial}{\left(z^2+2\right)^3}-4\,\frac{44\,z^2+16+42\,z^4+7\,z^6}{\left(z^2+2\right)^3}.$$

Removing apparent singularities of first-order systems

A new algorithm

Classification of Singularities

Consider a System of first order linear differential equations:

$$[A] \qquad \partial X = A(z)X, \;\; A(z) \in Mat_n(\mathbb{C}(z))$$

- A pole z_0 of $A(z)$ is a regular singular point for $[A]$ if there is a fund soln matrix W of $[A]$ has the form:

$$W(z) = \Phi(z)(z - z_0)^{\Lambda}$$

  where $\Phi(z)$ is holomorphic and Λ is a constant matrix.
- Otherwise z_0 is called an irregular singular point.
- A system [A] has regular singularity at z_0 iff it is gauge equivalent to a system [B] with a simple pole at z_0.

Desingularization of a First Order System

Consider a system

$$[A] \qquad \frac{d}{dz}X = A(z)X.$$

Def. A system

$$[B] \qquad \frac{d}{dz}Y = B(z)Y$$

with $B \in \mathbb{C}(z)^{n\times n}$ is called a desingularization of $[A]$ if:

(i) there exits a polynomial matrix $T(z)$ with $\det T(z) \not\equiv 0$ such that $B = T[A] := T^{-1}AT - T^{-1}T'$,

(ii) The singularities of $[B]$ are the singularities of $[A]$ that are not apparent.

Existence of Desingularization

Prop.0 If $z = z_0$ is a finite apparent singularity of $[A]$ then there exists a polynomial matrix $T(z)$ with

$$\det T(z) = c(z - z_0)^{\alpha}, \quad c \in \mathbb{C}^*, \alpha \in \mathbb{N}$$

such that $[B] := T[A]$ has no pole at $z = z_0$.

Proof.

- Every fund soln matrix Φ of $[A]$ is holomorphic (in a neighborhood of z_0);
- Since $\mathbb{C}[[z - z_0]]$ is a PID, there exists unimodular matrices $P(z) \in GL_n(\mathbb{C}[z])$, and $Q(z) \in GL_n(\mathbb{C}[[z - z_0]])$ such that

  $$P(z)\Phi(z)Q(z) = Diag((z - z_0)^{\alpha_1}, \ldots, (z - z_0)^{\alpha_n})$$

  where $\alpha_1, \ldots \alpha_n$ are nonnegative integers.
- Take

  $$T(z) \;=\; P^{-1}(z)\; Diag((z - z_0)^{\alpha_1}, \ldots, (z - z_0)^{\alpha_n})$$

How to detect and remove an apparent singularity?

Prop.1: If $z = z_0$ is a finite apparent singularity of $[A]$ then one can construct a polynomial matrix $T(z)$ with $\det T(z) = c(z - z_0)^\alpha$, $c \in \mathbb{C}^*$ and $\alpha \in \mathbb{N}$ such that $T[A]$ has at worst a simple pole at $z = z_0$.

This follows from the fact that:

- if z_0 is an apparent singularity then z_0 is a regular singularity,
- and that a system with a regular singularity at z_0 is equivalent to a system with a simple pole at z_0.

• The transformation T can be constructed using the *rational Moser algorithm* (developed in Bar'1995).

An example: Let $A \in M_4(\mathbb{Q}(z))$

$$A = \begin{bmatrix} \frac{-1}{z} & \frac{-1+z^{17}-8\,z^{14}+24\,z^{11}-32\,z^{8}+16\,z^{5}}{z^{3}(z^{3}-2)^{2}} & 0 & \frac{1}{z^{3}(z^{3}-2)^{2}} \\ 0 & 0 & \frac{1}{z^{3}(z^{3}-2)^{2}} & 0 \\ \frac{1}{(z^{3}-2)^{3}z^{4}} & 0 & \frac{4}{z} & 0 \\ 0 & -\frac{2}{z} & \frac{1}{z^{3}(z^{3}-2)^{2}} & \frac{2}{z} \end{bmatrix}$$

- Here the denominator of A is the polynomial $z^4(z^3-2)^3$.
- Take $p = z^3 - 2$. The algorithm in [Bar'95] produces the following transformation

$$T = \begin{bmatrix} 0 & \left(z^3-2\right)^3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & z^3-2 & 0 \\ \left(z^3-2\right)^4 & 0 & 0 & 1 \end{bmatrix}$$

and the equivalent matrix $B = T[A]$:

$$B = T[A] = \begin{bmatrix} -\frac{10\,z^3+4}{z(z^3-2)} & 0 & 0 & 0 \\ \frac{1}{z^3(z^3-2)} & -\frac{8\,z^3+2}{z(z^3-2)} & 0 & \frac{z^2}{z^3-2} \\ 0 & \frac{1}{z^4(z^3-2)} & \frac{z^3-8}{z(z^3-2)} & 0 \\ 0 & 0 & \frac{1}{z^3(z^3-2)} & 0 \end{bmatrix}$$

- the denominator of the matrix B is $z^4(z^3-2)$.
- Hence the differential system $[A]$ has regular singularities at the zeros of z^3-2.

The same algorithm applied to B and the point $z = 0$ produces the transformation

$$S = \begin{bmatrix} 0 & 0 & 0 & -z^5 \\ 0 & z^4 & 0 & 0 \\ 0 & 0 & z^2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

and the equivalent matrix

$$C = S[B] = \begin{bmatrix} 0 & 0 & \frac{1}{z(z^3-2)} & \frac{15\,z^3-6}{z(z^3-2)} \\ \frac{1}{z^2(z^3-2)} & -\frac{12\,z^3-6}{z(z^3-2)} & 0 & 0 \\ 0 & \frac{1}{z^2(z^3-2)} & -\frac{4+z^3}{z(z^3-2)} & 0 \\ 0 & 0 & 0 & -\frac{15\,z^3-6}{z(z^3-2)} \end{bmatrix}$$

Hence the point $z = 0$ is an irregular singular point of the original system $[A]$.

Prop.2 Suppose that $A(z)$ has simple pole at $z = z_0$ and let

$$A(z) = \frac{A_0}{(z - z_0)} + \sum_{k \geq 1} A_k (z - z_0)^{k-1}, \;\; A_k \in \mathbb{C}^{n \times n}.$$

If z_0 is an apparent singularity then the eigenvalues of A_0 are nonnegative integers

Prop.3: Suppose that $z = z_0$ is a simple pole of $A(z)$ and that its residue matrix A_0 has only nonnegative integer eigenvalues. Then one can construct a polynomial matrix $T(z)$ with

$$\det T(z) = c(z - z_0)^{\alpha}$$

for some $c \in \mathbb{C}^*$ and $\alpha \in \mathbb{N}$ such that

$$B := T[A] = B_0(z - z_0)^{-1} + \cdots$$

has at worst a simple pole at $z = z_0$ with

$$B_0 = mI_n + N$$

where $m \in \mathbb{N}$ and N nilpotent.

- Moreover z_0 is an apparent singularity iff $N = 0$.
- In this case the gauge transformation $Y = (z - z_0)^m \tilde{Y}$ leads to a system for which $z = z_0$ is an ordinary point.

Main idea of the proof:

- The eigenvalues of A_0 of are nonnegative integers:

$$m_1 > m_2 > \ldots > m_s, \quad m_i - m_{i+1} = \ell_i \in \mathbb{N}^*, \quad i = 1, \ldots, s-1.$$

- By applying a constant gauge transformation we can assume that:

$$A_0 = \begin{pmatrix} A_0^{11} & 0 \\ 0 & A_0^{22} \end{pmatrix},$$

where A_0^{11} is an ν_1 by ν_1 matrix having one single eigenvalue m_1

$$A_0^{11} = m_1 I_{\nu_1} + N_1$$

N_1 being a nilpotent matrix.

• Apply the gauge transformation $U = diag((z-z_0)I_{\nu_1}, I_{n-\nu_1})$ yields the new system:

$$Z' = (z-z_0)^{-1}\tilde{A}(z)Z, \quad \tilde{A}(z) = (z-z_0)U^{-1}A(z)U - (z-z_0)U^{-1}U'$$

with the leading matrix:

$$\tilde{A}(0) = \left(A_0 + (z-z_0)U^{-1}A_1U - (z-z_0)U^{-1}U'\right)_{|z=z_0}.$$

• Let A_1 be partitioned as A_0 :

$$A_1 = \begin{pmatrix} A_1^{11} & A_1^{12} \\ A_1^{21} & A_1^{22} \end{pmatrix}, \quad A_1^{11} \in \mathbb{C}^{\nu_1 \times \nu_1}$$

Then

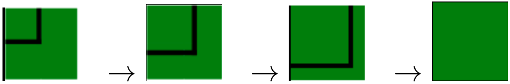
$$\tilde{A}(0) = \begin{pmatrix} A_0^{11} - I_{\nu_1} & A_1^{12} \\ 0 & A_0^{22} \end{pmatrix}.$$

Hence the eigenvalues of $\tilde{A}(0)$ are: $m_1 - 1, m_2, \ldots, m_s$, each with the same initial multiplicity ν_i.

- By repeating this process ℓ_1 times, the eigenvalues become:

$$m_1 - \ell_1 = m_2, m_2, \ldots, m_s.$$

- Thus, after $\ell_1 + \ldots + \ell_{s-1}$ steps, the eigenvalues $m_1, \ldots, m_s$ are reduced to one single eigenvalue m_s of multiplicity $\nu_1 + \ldots + \nu_s = n$.

$A_0 =$  $= m_s I_n + N$

- z_0 is an apparent singularity iff $N = 0$.

- In this case the gauge transformation $Y = (z - z_0)^{m_s} \tilde{Y}$ leads to a system for which $z = z_0$ is an ordinary point.

• The matrix T in Prop3 is obtained as a product of invertible constant matrices or diagonal matrices of the form $U = diag((z - z_0)I_\nu, I_{n-\nu})$.

Hence T is a polynomial matrix with $\det T(z) = c(z - z_0)^\alpha$ for some $c \in \mathbb{C}$ and $\alpha \in \mathbb{N}$.

• Due to the form of its determinant, the gauge transformation $T(z)$ in the above proposition does not affect the other finite singularities of $[A]$. We have:

Theorem One can construct a polynomial matrix $T(z)$ which is invertible in $\mathbb{C}(z)$ such that the finite poles of $B := T[A]$ are exactly the real singularities for $[A]$.

Algorithm

1. Let $\mathcal{P}(A)$ be the set of poles of A.
2. Compute a polynomial matrix $T(z)$ such that
   - the zeros of $\det T(z)$ are in $\mathcal{P}(A)$
   - $T[A]$ has the same poles as A with minimal orders.
3. For each simple pole z_0 compute A_{0,z_0} the residue matrix of $A(z)$ at $z = z_0$ and its eigenvalues.
4. Let $\mathcal{A}pp(A)$ denote the set of singularities z_0 such that A_{0,z_0} has only nonnegative integer eigenvalues.
5. For each $z_0 \in \mathcal{A}pp(A)$ compute a polynomial matrix $T_{z_0}(z)$ with $\det T_{z_0}(z) = c(z - z_0)^{\alpha}$ such that $T_{z_0}[A]$ has at worst a simple pole at $z = z_0$ with residue matrix of the form $R_{z_0} = m_{z_0} I_n + N_{z_0}$ where $m_{z_0} \in \mathbb{N}$ and N_{z_0} nilpotent.
6. Keep in $\mathcal{A}pp(A)$ only the points z_0 for which $N_{z_0} = 0$.
7. The scalar transformation $T = \prod_{z_0 \in \mathcal{A}pp(A)} (z - z_0)^{m_{z_0}} I_n$ yields a desingularization of the original system $[A]$.

Application to Desingularization of Scalar Differential Equations

Example 5

- Let $\partial = \frac{d}{dz}$ and consider

$$L = \partial^2 - \frac{(z^2-3)\,(z^2-2\,z+2)}{(z-1)\,(z^2-3\,z+3)\,z}\partial + \frac{(z-2)\,(2\,z^2-3\,z+3)}{(z-1)(z^2-3\,z+3)z}.$$

- L has apparent singularities at $z = 0$ and the roots of $z^2 - 3\,z + 3 = 0$.

- A desingularization computed by the classical algorithm[†] is given by:

$$\begin{aligned} \tilde{L}_{Classical} \quad = \quad & (z-1)\,(z^4 - z^3 + 3\,z^2 - 6\,z + 6)\partial^4 \\ & -(z^5 - 2\,z^4 + z^3 - 12\,z^2 + 24\,z - 24)\,\partial^3 \\ & -(3\,z^3 + 9\,z^2)\,\partial^2 + (6\,z^2 + 18\,z)\,\partial - (6\,z + 18). \end{aligned}$$

[†] Exm 1, Chen-Kauers-Singer'14

- A desingularization computed by the probabilistic method of CKS14[‡] is given by:

$$\begin{aligned}\tilde{L}_{CKS} &= (z-1)\,(z^6 - 3\,z^5 + 3\,z^4 - z^3 + 6)\,\partial^4 \\ &\quad - (2\,z^6 - 9\,z^5 + 15\,z^4 - 11\,z^3 + 3\,z^2 - 24)\,\partial^3 \\ &\quad - (z^7 - 4\,z^6 + 6\,z^5 - 4\,z^4 + z^3 + 6\,z - 6)\,\partial \\ &\quad + (2\,z^6 - 9\,z^5 + 15\,z^4 - 11\,z^3 + 3\,z^2 - 24).\end{aligned}$$

- The removal of one apparent singularity introduces new singularities. The latter can then be removed by using a trick introduced in ABH algorithm.

[‡] Exm 7(1), Chen-Kauers-Singer'14

- The desingularization computed by ABH method is:

$$
\begin{aligned}
\tilde{L}_{ABH} \quad = \quad & \partial^4 + \frac{(16\,z^4 - 55\,z^3 + 63\,z^2 - 42\,z + 36)}{9\,(z-1)}\,\partial^3 \\
& - \frac{(64\,z^5 - 316\,z^4 + 591\,z^3 - 468\,z^2 + 123\,z + 42)}{9\,(z-1)^2}\,\partial^2 \\
& - \frac{96\,z^5 - 570\,z^4 + 1333\,z^3 - 1597\,z^2 + 993\,z - 219}{9\,(z-1)^3} \\
& + \frac{\beta}{9\,(z-1)^3}\,\partial,
\end{aligned}
$$

where

$$
\beta = (48\,z^6 - 197\,z^5 + 148\,z^4 + 488\,z^3 - 1162\,z^2 + 999\,z - 288).
$$

- The companion matrix of L is

$$A = \begin{bmatrix} 0 & 1 \\ \frac{(z-2)\,(2\,z^2-3\,z+3)}{(z-1)\,(z^2-3\,z+3)\,z} & \frac{(z^2-3)\,(z^2-2\,z+2)}{(z-1)\,(z^2-3\,z+3)\,z} \end{bmatrix}$$

- Our new algorithm computes the following gauge transformation T

$$T = \begin{bmatrix} 1 & 0 \\ 1 & (-z^2+3\,z-3)\,z^2 \end{bmatrix}$$

- The matrix of the new equivalent system is

$$B = T^{-1}(AT - T') = \begin{bmatrix} 1 & -z^2\,(z^2-3\,z+3) \\ 0 & \frac{2}{1-z} \end{bmatrix}$$

- It has $z = 0$ and roots of $z^2 - 3\,z + 3 = 0$ as ordinary points.
- No new apparent singularities are introduced.

Comments

- The desingularization algorithms developed specifically for scalar equations are based on computing a least common left multiple of the operator in question and an appropriately chosen operator.
- This outputs an equation whose solution space contains strictly the solution space of the input equation.
- The new algorithm is based on an adequate choice of a gauge transformation.
- The desingularized output system is always equivalent to the input system and the dimension of the solution space is preserved.
- The transformations and the equivalent systems computed by our algorithm, have rational function coefficients.

Summary

- We gave a method for detecting and removing the apparent singularities of linear differential systems via a rational algorithm, i.e. an algorithm which avoids the computations with individual conjugate singularities.
- Our method can be used, in particular, for the desingularization of differential operators in the scalar case.
- Maple Package available for download at:

  $http://www.unilim.fr/pages\_perso/suzy.maddah/Research.html$

- More examples can be found there:
  - Desingularization at polynomial of degree 4: The Ising Model[§] in statistical physics.
  - Desingularization at polynomial of degree 37.

[§] Bostan-Boukraa-Hassani-van Hoeij-Maillard-Weil-Zenine

Further investigations

- The complexity study of the various algorithms existing for the scalar case, as well as this new algorithm which can be applied to the companion system, so that their efficiency can be compared (work in progress).
- Investigating the case of difference systems.