Computer algebra methods for testing the structural stability of multidimensional systems

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Overview

Overview

Discrete-time linear shift-invariant systems

- Let $(y_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ be 2 sequences satisfying the equation:

$$\begin{cases} y_{n+2} - 3\,y_{n+1} + 2\,y_n = 2\,u_{n+1} + 2\,u_n, \\ y_0 = 0, \\ y_1 = 0. \end{cases} \qquad (\star)$$

- **Definition:** The $\mathcal{Z}$-transform of a sequence $= (x_n)_{n\in\mathbb{Z}}$ is defined by:

$$\mathcal{Z}((x_n)_{n\in\mathbb{Z}})(z) := \sum_{n\in\mathbb{Z}} x_n\, z^{-n}.$$

$$(\star) \Rightarrow \ \mathcal{Z}(y)(z) = \frac{2\,(z^{-1}+1)}{z^{-2} - 3\,z^{-1} + 2}\,\mathcal{Z}(u)(z) = \frac{2\,(z^2+z)}{2\,z^2 - 3\,z + 1}\,\mathcal{Z}(u)(z).$$

- **Definition:** The rational function

$$P(z) := \frac{\sum_{j=0}^{s} b_j\, z^{-j}}{\sum_{i=0}^{r} a_i\, z^{-i}}$$

is called the transfer function of the discrete-time linear system:

$$\sum_{i=0}^{r} a_i\, y_i = \sum_{j=0}^{s} b_j\, u_j.$$

Discrete multidimensional systems

- Roesser model:

$$\begin{cases} \begin{pmatrix} x_h(i+1,j) \\ x_v(i,j+1) \end{pmatrix} = A \begin{pmatrix} x_h(i,j) \\ x_v(i,j) \end{pmatrix} + B\, u(i,j), \\ y(i,j) = C \begin{pmatrix} x_h(i,j) \\ x_v(i,j) \end{pmatrix} + D\, u(i,j). \end{cases}$$

- Fornasini-Marchesini models:

$$x(i+1,j+1) = A_1\, x(i+1,j) + A_2\, x(i,j+1) + B_1\, u(i+1,j) + B_2\, u(i,j+1), \ldots$$

- *n*-dimensional recursive filters: digitial image processing, ...

$$\mathbf{j} := (j_1, \ldots, j_n),\ \mathbf{k} := (k_1, \ldots, k_n) \in \mathbb{Z}^n \ \Rightarrow\ \mathbf{k} - \mathbf{j} := (k_1 - j_1, \ldots, k_n - j_n)$$

$$\mathcal{Z}((h(\mathbf{k}))_{\mathbf{k}\in\mathbb{Z}^n})(z) := \sum_{\mathbf{k}\in\mathbb{Z}^n} h(\mathbf{k})\, z^{-\mathbf{k}}, \quad z^{-\mathbf{k}} := z_1^{-k_1} \ldots z_n^{-k_n}.$$

$$y(\mathbf{k}) = (h \star u)(\mathbf{k}) := \sum_{\mathbf{j}\in\mathbb{Z}^n} h(\mathbf{k} - \mathbf{j})\, u(\mathbf{j}) \ \Rightarrow\ \mathcal{Z}(y)(z) = \mathcal{Z}(h)(z)\, \mathcal{Z}(u)(z),$$

$$\mathcal{Z}(h)(z_1, \ldots, z_n) = \frac{n(z_1, \ldots, z_n)}{d(z_1, \ldots, z_n)} \in \mathbb{R}(z_1, \ldots, z_n).$$

Overview

Structural stability

- $P := \frac{N(z_1,\ldots,z_n)}{D(z_1,\ldots,z_n)} \in \mathbb{R}(z_1,\ldots,z_n)$: a transfer function ($\gcd(D,N)=1$).

- The closed unit polydisc of $\mathbb{C}^n$:

$$\overline{\mathbb{D}}^n := \{z = (z_1,\ldots,z_n) \in \mathbb{C}^n \mid |z_i| \leq 1,\ i = 1,\ldots,n,\ \}.$$

- **Definition:** P is structurally stable if D is devoid from zero in $\overline{\mathbb{D}}^n$, i.e.:

$$\forall\ z = (z_1,\ldots,z_n) \in \overline{\mathbb{D}}^n:\ \ D(z_1,\ldots,z_n) \neq 0. \qquad (1)$$

- The affine algebraic set associated to D:

$$V_{\mathbb{C}}(D) := \{z = (z_1,\ldots,z_n) \in \mathbb{C}^n \mid D(z_1,\ldots,z_n) = 0\}.$$

- Condition (1) is equivalent to:

$$V_{\mathbb{C}}(D) \cap \overline{\mathbb{D}}^n = \emptyset.$$

Previous works

The case $n = 1$

- Check that the complex roots of the polynomial

$$D(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$$

do not belong to $\mathbb{D} := \{z \in \mathbb{C} \mid |z| \leq 1\}$.

- Several algebraic stability criteria (Jury test, Bistritz test, etc), discrete time analogues of the Routh-Hurwitz criterion.

- Based on Cauchy index computation: sign variation in some polynomial sequences.

- The complexity of a univariate gcd computation.

Previous works

Bistritz test

- Let $D(z) := a_n\, z^n + a_{n-1}\, z^{n-1} + \ldots + a_0$ and $D^\star(z) := z^n\, D(z^{-1})$.
- Compute the sequence of polynomials $\{T_i(z)\}_{i=n,\ldots,0}$, defined by

$$\begin{cases} T_n(z) := D(z) + D^\star(z), \\ T_{n-1}(z) := \dfrac{D(z) + D^\star(z)}{(z-1)}, \\ T_{i-1}(z) := \dfrac{\delta_{i+1}(1+z)\,T_i(z) - T_{i+1}(z)}{z}, \end{cases}$$

where $\delta_{i+1} := \frac{T_{i+1}(0)}{T_i(0)}$ for $i = n-1, \ldots, 1$.

- **Criterion:** The system is stable if and only if the sequence is normal and the number of sign variation in $\{T_n(1), \ldots, T_0(1)\}$ is zero.

Previous works

The case n>1

- **First step:** Simplification of the initial condition.
- [Strintzis,Huang 1977]:

$$\begin{cases} D(0,\ldots,0,z_n) \neq 0, & |z_n| \leq 1, \\ D(0,\ldots,0,z_{n-1},z_n) \neq 0, & |z_{n-1}| \leq 1,\ |z_n| = 1, \\ \vdots & \vdots \\ D(0,z_2,\ldots,z_n) \neq 0, & |z_2| \leq 1,\ |z_i| = 1, i > 2, \\ D(z_1,z_2,\ldots,z_n) \neq 0, & |z_1| \leq 1,\ |z_i| = 1, i > 1. \end{cases}$$

- [DeCarlo et al, 1977]:

$$\begin{cases} D(z_1,1,\ldots,1) \neq 0, & |z_1| \leq 1, \\ D(1,z_2,1,\ldots,1) \neq 0, & |z_2| \leq 1, \\ \vdots & \vdots \\ D(1,\ldots,1,z_n) \neq 0, & |z_n| \leq 1, \\ D(z_1,\ldots,z_n) \neq 0, & |z_1| = \ldots = |z_n| = 1. \end{cases}$$

Previous works

The case n>1

- **Second step:** Implementation.
- **The case $n = 2$:** Numerous tests (Bistritz (94,99,02,03,04), Xu et al. 04, Fu et al. 06, etc).
  - Most of them are based on Strintzis conditions.
  - Generalization of univariate tests (sub-resultant computation).
- **The case $n > 2$:** very few (Dumetriscu 06, Serban and Najim 07).
  - Sum of square techniques: either inefficient or conservative.

Overview

DeCarlo's conditions

- Start with DeCarlo's conditions:

$$\begin{cases} D(z_1, 1, \ldots, 1) \neq 0, & |z_1| \leq 1, \\ D(1, z_2, 1, \ldots, 1) \neq 0, & |z_2| \leq 1, \\ \vdots & \vdots \\ D(1, \ldots, 1, z_n) \neq 0, & |z_n| \leq 1, \\ D(z_1, \ldots, z_n) \neq 0, & |z_1| = \ldots = |z_n| = 1. \end{cases}$$

- All the conditions except the last one can be tested using classical univariate stability tests.

- **Focus on the condition:** $D(z_1, \ldots, z_n) \neq 0,\ |z_1| = \ldots = |z_n| = 1$.

One first approach

• $z_k := x_k + i\, y_k, \quad x_k,\, y_k \in \mathbb{R}, \quad k = 1, \ldots, n, \quad i^2 = -1.$

The problem is equivalent to the study of the algebraic system:

$$(S) \left\{ \begin{array}{l} \mathcal{R}(D(x_1 + i\, y_1, \ldots, x_n + i\, y_n)) := \mathcal{R}(x_1, y_1, \ldots, x_n, y_n) = 0, \\ \mathcal{C}(D(x_1 + i\, y_1, \ldots, x_n + i\, y_n)) := \mathcal{C}(x_1, y_1, \ldots, x_n, y_n) = 0, \\ x_1^2 + y_1^2 - 1 = 0, \\ \qquad \vdots \\ x_n^2 + y_n^2 - 1 = 0. \end{array} \right.$$

• **Case** $n = 2$: zero-dimensional system $\rightsquigarrow$ univariate rational representation, triangular representation, Gröbner bases.

• **Case** $n > 2$: systems with positive dimension $\rightsquigarrow$ cylindrical algebraic decomposition, critical points methods

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- **Case** $n = 2$: zero-dimensional system $\rightsquigarrow$ univariate rational representation, triangular representation, Gröbner bases.

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Drawback: The number of variables is doubled!

Alternative approach

- **Goal:** Avoid doubling the number of variables.

- The unit poly-circle defines a n-D subspace in the $2n$-D complex space.

- Via some transformations, the problem can be reduced to the search of zeros in the real space $\mathbb{R}^n$.

- The obtained conditions are checked using classical algorithms for solving algebraic systems.

Unit circle parametrization

- Only the zeros of D on the unit poly-circle $\mathbb{T}^n$ are considered.

- Use the parametrization of the complex unit circle:

$$\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}.$$

- $z_k := \frac{(1-x_k^2)}{(1+x_k^2)} + i\, \frac{2\,x_k}{(1+x_k^2)}, \quad k = 1, \ldots, n$

- Let $\mathcal{R}(x_1, \ldots, x_n) + i\,\mathcal{C}(x_1, \ldots, x_n)$ be the numerator of the fraction:

$$D\left(\frac{1-x_1^2}{1+x_1^2} + i\,\frac{2x_1}{1+x_1^2}, \ldots, \frac{1-x_n^2}{1+x_n^2} + i\,\frac{2x_n}{1+x_n^2}\right).$$

- **Theorem:** $\mathcal{V}_{\mathbb{C}}(D) \cap [\mathbb{T} \setminus \{-1\}]^n = \emptyset \iff \mathcal{V}_{\mathbb{R}}(\mathcal{R}, \mathcal{C}) = \emptyset.$

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- **Theorem:** $\mathcal{V}_{\mathbb{C}}(D) \cap [\mathbb{T} \setminus \{-1\}]^n = \emptyset \iff \mathcal{V}_{\mathbb{R}}(\mathcal{R}, \mathcal{C}) = \emptyset.$

Drawback: The degree is doubled!

Möbius transformation

- **Definition:** A Möbius transformation is a rational function

$$\begin{array}{rcl}\phi : \overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\} & \longrightarrow & \overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \\ z & \longmapsto & \frac{a\,z+b}{c\,z+d},\end{array}$$

for $a, b, c, d \in \mathbb{C}$ satisfying $ad - bc \neq 0$ $\left(\phi\left(-\frac{d}{c}\right) = \infty,\ \phi(\infty) = \frac{a}{c}\right)$.

- The Möbius transformation $\phi(z) := \frac{z-i}{z+i}$ maps the real line $\overline{\mathbb{R}} := \mathbb{R} \cup \infty$ to the unit complex circle $\mathbb{T}$.

  - $z_k := \frac{(x_k - i)}{(x_k + i)}, \quad k = 1, \ldots, n.$

- Let $\mathcal{R}(x_1, \ldots, x_n) + i\,\mathcal{C}(x_1, \ldots, x_n)$ be the numerator of the fraction:

$$D\left(\frac{x_1 - i}{x_1 + i}, \ldots, \frac{x_n - i}{x_n + i}\right).$$

- **Theorem:** $\mathcal{V}_{\mathbb{C}}(D) \cap [\mathbb{T} \setminus \{1\}]^n = \emptyset \iff \mathcal{V}_{\mathbb{R}}(\mathcal{R}, \mathcal{C}) = \emptyset.$

- **Remark:** The total degree of $\mathcal{R}$ and $\mathcal{C}$ is bounded by $\sum_{i=1}^{n} deg_{z_i}(D)$

Checking for real zeros in $\mathbb{R}^n$

Critical point methods

- Basu, Pollack, Roy, Giusti, Heintz, Bank, Safey El Din ...

- Principle: Computation of the critical points of some polynomial application π restricted to the algebraic set $\mathcal{V} := V(\langle \mathcal{R}, \mathcal{C} \rangle)$.

- Algorithm: Based on Safey El Din and Schost 2003

- Compute zero-dimensional systems that encode these critical points and check if they admits real solutions.

- A convenient representation of the solutions is the Rational Univariate Representation (RUR).

Rational Univariate Representation

- Let $I \subset \mathbb{R}[x_1, \ldots, x_n]$ be a zero-dimensional ideal and $V(I) \subset \mathbb{C}^n$ its variety.

A RUR of I is given by:

- A linear form $a_1 x_1 + \ldots + a_n x_n$ that separates the points of V:
- A one-to-one mapping between the roots of an univariate polynomial f and the solutions of V:

$$
\begin{array}{cccc}
\phi_t : & V_{\mathbb{C}}(I) & \approx & V_{\mathbb{C}}(f) \\
& \alpha & \longmapsto & t(\alpha), \\
& \left(\dfrac{f_{x_1}(\beta)}{f_1(\beta)}, \ldots, \dfrac{f_{x_n}(\beta)}{f_1(\beta)} \right) & \longleftarrow & \beta.
\end{array}
$$

- $V(I) \cap \mathbb{R}^n = \emptyset$ if and only if $V(f) \cap \mathbb{R} = \emptyset \rightsquigarrow$ Sturm sequence.

The overall algorithm

Procedure: IsStable

```
begin
    Data : D(z_1, ..., z_n) ∈ R[z_1, ..., z_n]
    Result : return True if V(D(z_1, ..., z_n)) ∩ 𝔻^n = ∅
    for k = 0 to n − 2 do
        Compute S_k, the set of polynomials obtained from D(z_1, ..., z_n) after
        substituting k variables by 1
        foreach D_k in S_k do
            {R, C} = Möbius_transform(D_k)
            if V_ℝ({R, C}) ≠ ∅ then
                return False
            end
        end
    end
    if all the univariate polynomials in S_{n−1} are stable then
        return True
        else
            return false
        end
    end
end
```

Implementation

- A Maple procedure is provided based on:
  - The univariate stability test of Bistritz.
  - The external Maple package RAGlib for the study of real zeros of polynomial systems developed by M. Safey El Din.
- This implementation is able to check the stability of systems in 5 variables with moderate degree (5-10).

nb var \ degree		3	5	8	10	12
2	sparse	0.1	0.12	0.3	0.5	4
	dense	0.1	0.2	0.9	3.0	12
3	sparse	0.15	0.3	0.8	3	15
	dense	1	2	12	53	342

Table: CPU times in seconds of `IsStable` run on random polynomials in 2 and 3 variables with rational coefficients.

Overview

Ongoing work

• The first certified structural stability test for multidimensional systems.

• Stability of systems with parameters: compute regions of the parameter space where the system is structurally stable.

• Handle an arbitrary algebraic set instead of an hypersurface: needed for testing the stabilization.

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