Integer multiplication with generalized Fermat primes

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People believed long enough it was the best complexity we could reach (Kolmogorov). Karatsuba proved that it was wrong...

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Input: 2 numbers a and b of N bits.

We decompose the input into 2 polynomials $A = \sum_i a_i x^i$ and $B = \sum_i b_i x^i$ ($\deg A = \deg B = n$, $|a_i| = |b_i| = N/(2n) = k$, and $a_i = b_i = 0$ for $i > n$).

$$A(2^k) = a_0 + 2^k \times a_1 + \cdots + a_{2n-1} \times 2^{(2n-1)k} = a$$
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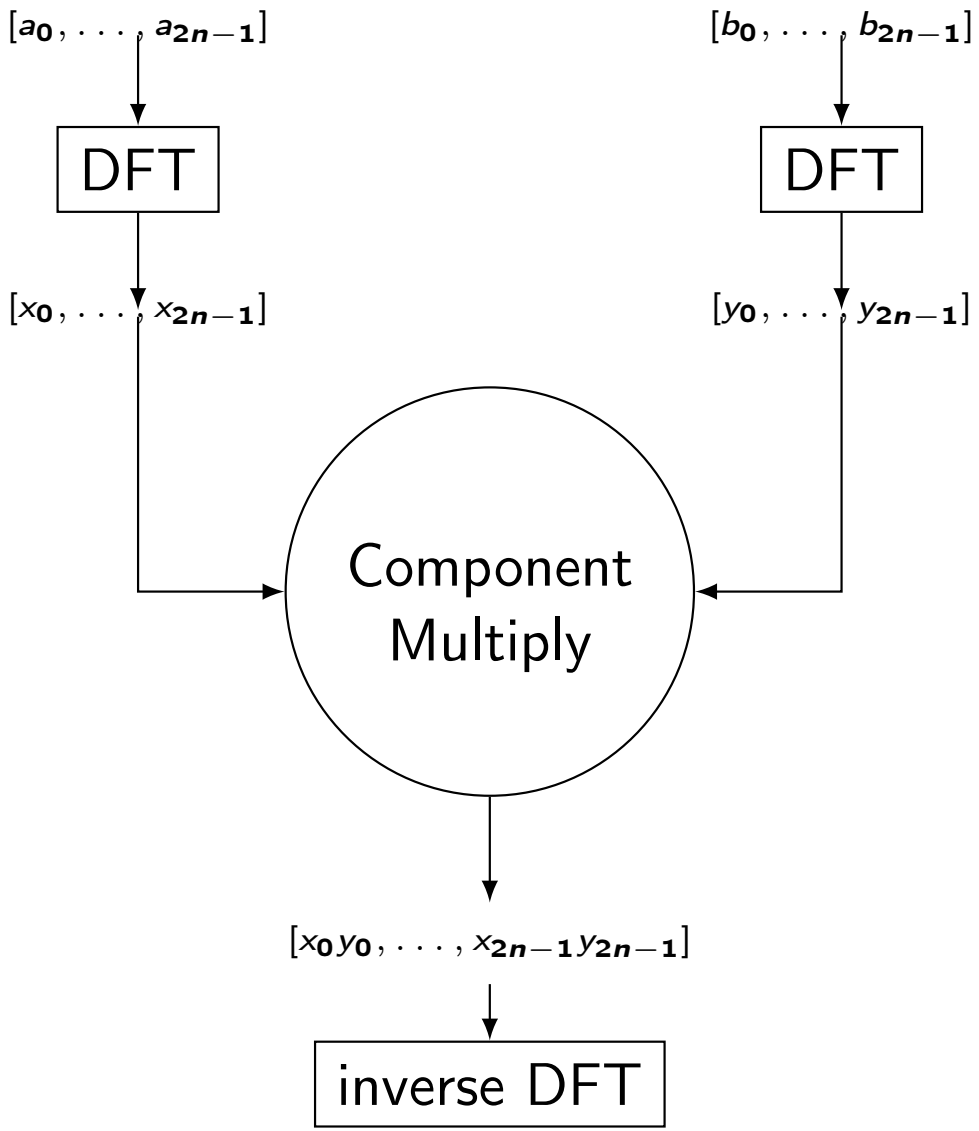
The DFT algorithm allows one to compute the $A(\omega^i)$ and $B(\omega^i)$.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inverse DFT

P is a polynomial of degree $2n-1$ (n is a power of 2) and ω is a $2n$-th principal root of unity in $\mathcal{R}$ ($\mathbb{C}$ or $\mathbb{Z}/p\mathbb{Z}$ for example), which means that $\sum_{j\in[0,2n-1]} \omega^{ij} = 0$ for $i \in [1, 2n-1]$.

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FFT$(P, \omega, 2n-1)$

---

if $n = 1$ **then**
  return $P_0 + P_1 + X(P_0 - P_1)$
end if
$P^{even} \leftarrow (P_{2i})_i$
$P^{odd} \leftarrow (P_{2i+1})_i$
$Q^{even} \leftarrow$ FFT$(P^{even}, \omega^2, n-1)$
$Q^{odd} \leftarrow$ FFT$(P^{odd}, \omega^2, n-1)$
$Q \leftarrow Q^{even}(X) + Q^{odd}(\omega X) + X^n \cdot (Q^{odd}(X) - Q^{even}(\omega X))$
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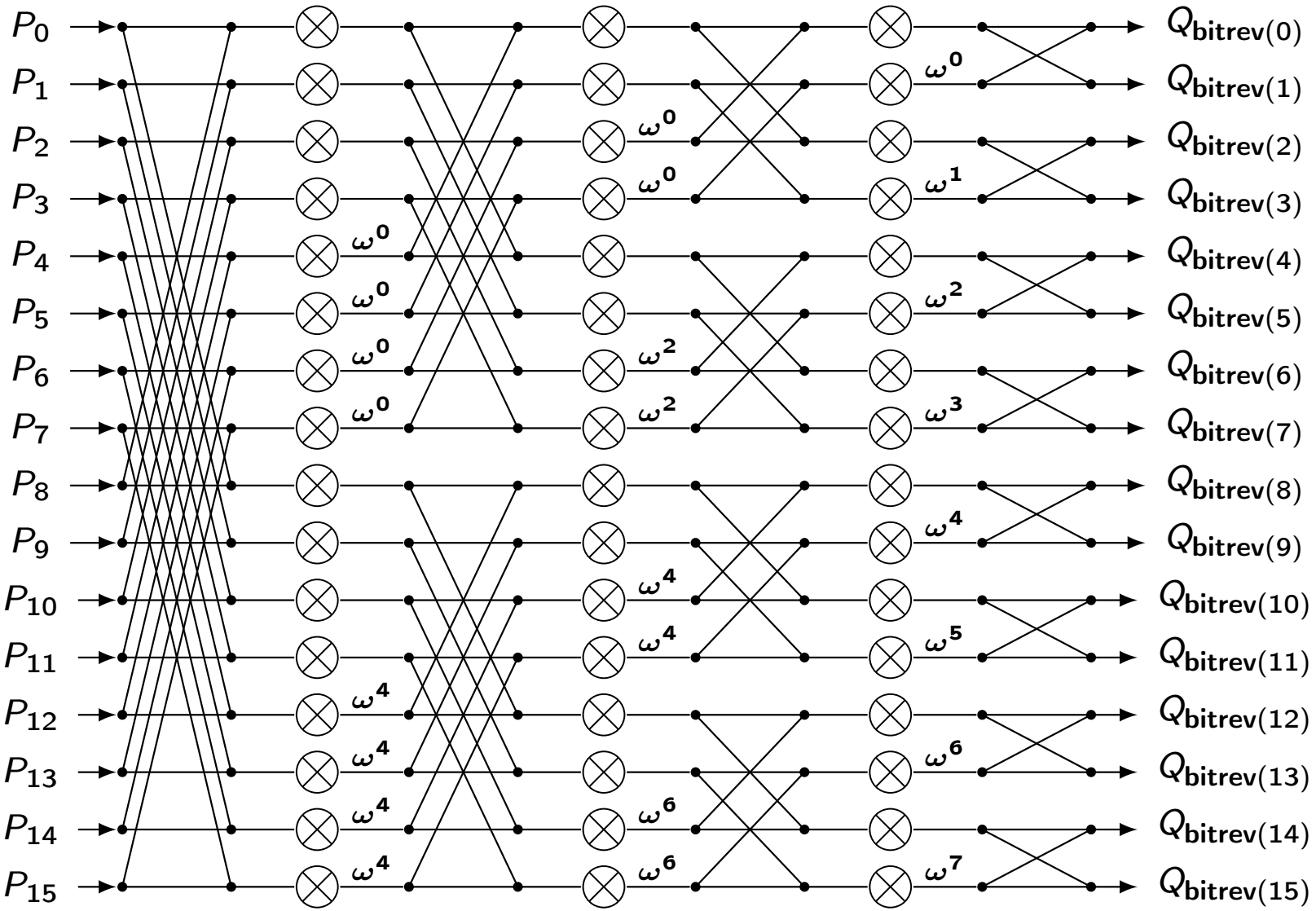
Complexity: $O(n \log n)$ operations in $\mathcal{R}$, among which multiplications by some powers of ω, additions and subtractions.


P_0
P_1
P_2
P_3
P_4
P_5
P_6
P_7
P_8
P_9
P_{10}
P_{11}
P_{12}
P_{13}
P_{14}
P_{15}
ω^0 ω^0 ω^0 ω^0
ω^4 ω^4 ω^4 ω^4
ω^0 ω^0 ω^2 ω^2
ω^4 ω^4 ω^6 ω^6
ω^0 ω^1 ω^2 ω^3 ω^4 ω^5 ω^6 ω^7
$Q_{\text{bitrev}(0)}$
$Q_{\text{bitrev}(1)}$
$Q_{\text{bitrev}(2)}$
$Q_{\text{bitrev}(3)}$
$Q_{\text{bitrev}(4)}$
$Q_{\text{bitrev}(5)}$
$Q_{\text{bitrev}(6)}$
$Q_{\text{bitrev}(7)}$
$Q_{\text{bitrev}(8)}$
$Q_{\text{bitrev}(9)}$
$Q_{\text{bitrev}(10)}$
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Examples:

- If $\mathcal{R} = \mathbb{C}$, then $\omega = \exp(i\pi/n)$. $k = O(\log N)$ is the best choice. Thus, the recursive calls are manipulating $O(\log N)$-sized data during convolution step.

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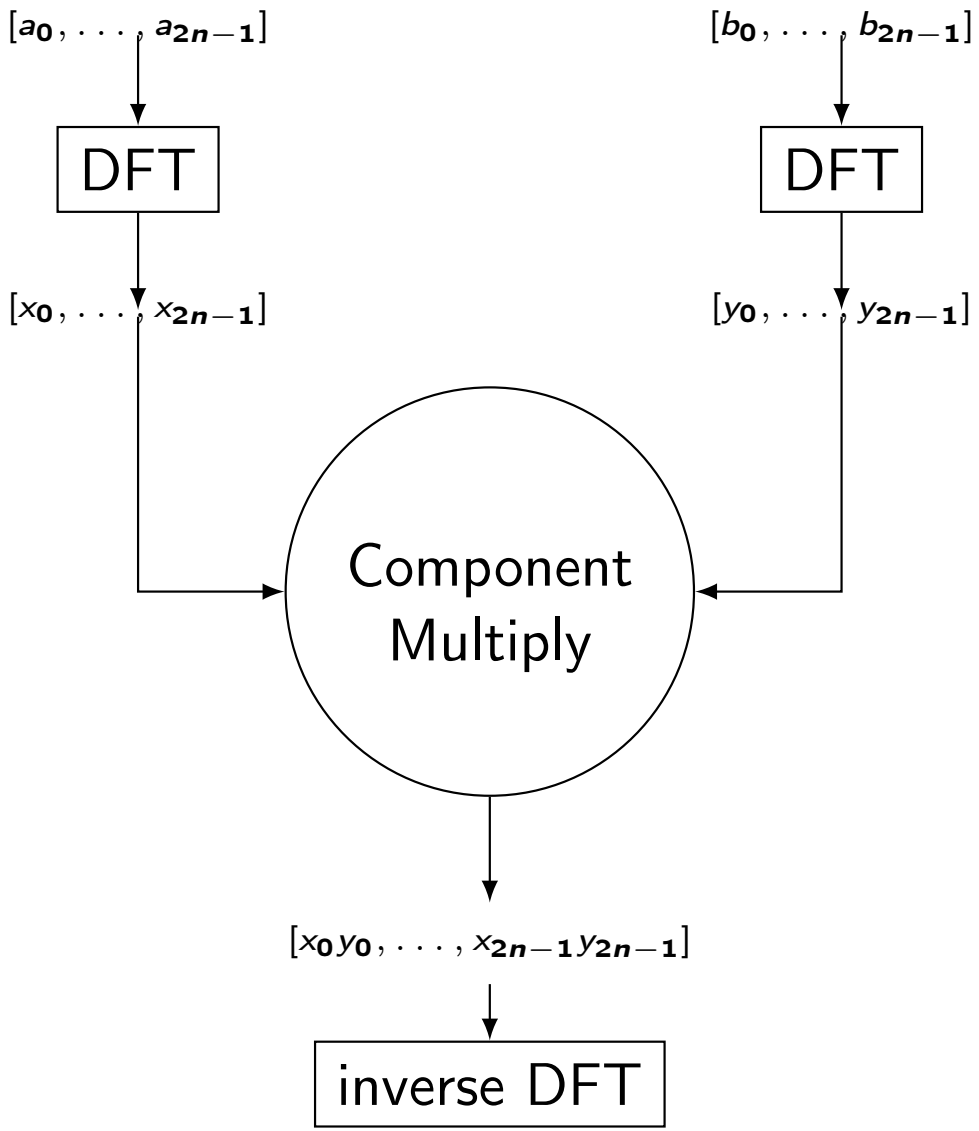
- If $\mathcal{R} = \mathbb{Z}/(2^e + 1)\mathbb{Z}$, then, $k \simeq e \simeq O(\sqrt{N})$ (smallest k possible).
  Then $\omega = 2^j$ with $j = e/n$. Multiplications by powers of ω are negacyclic permutations.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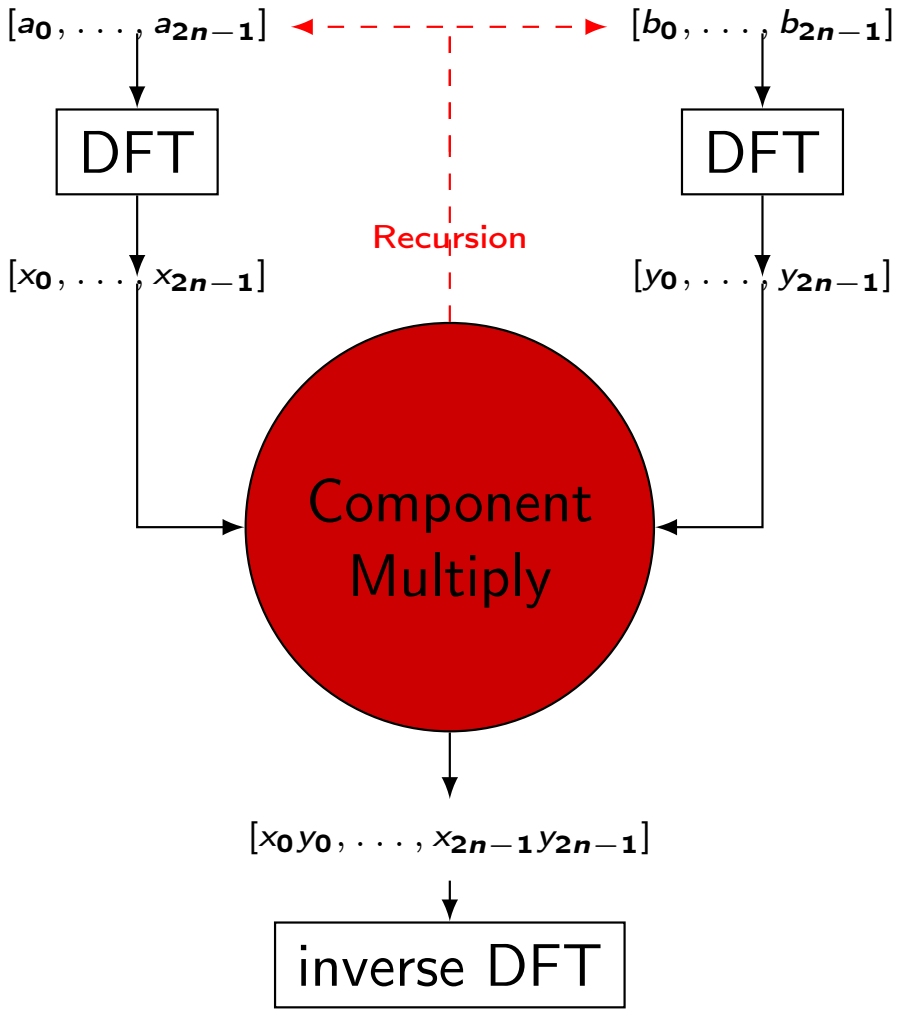
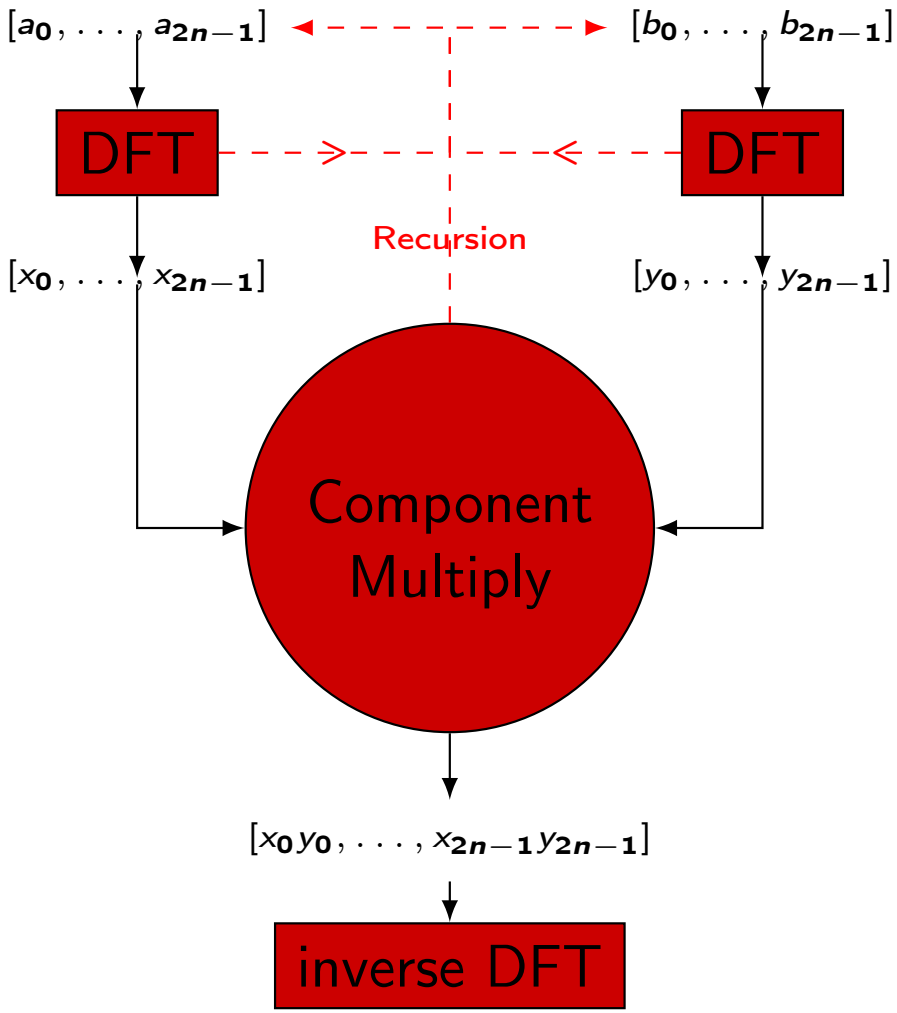
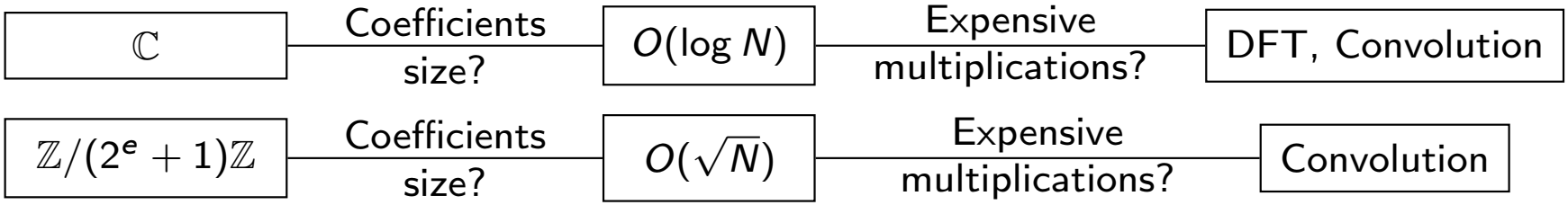
Modular Case



Complex Case




$\mathbb{C}$
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$O(\log N)$
Expensive multiplications?
DFT, Convolution
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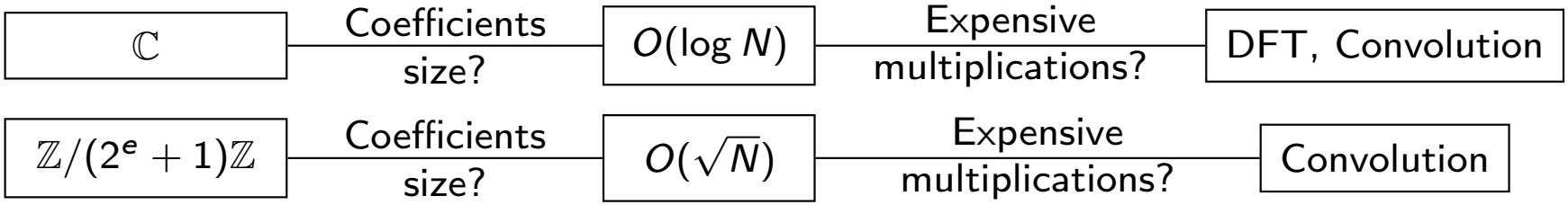

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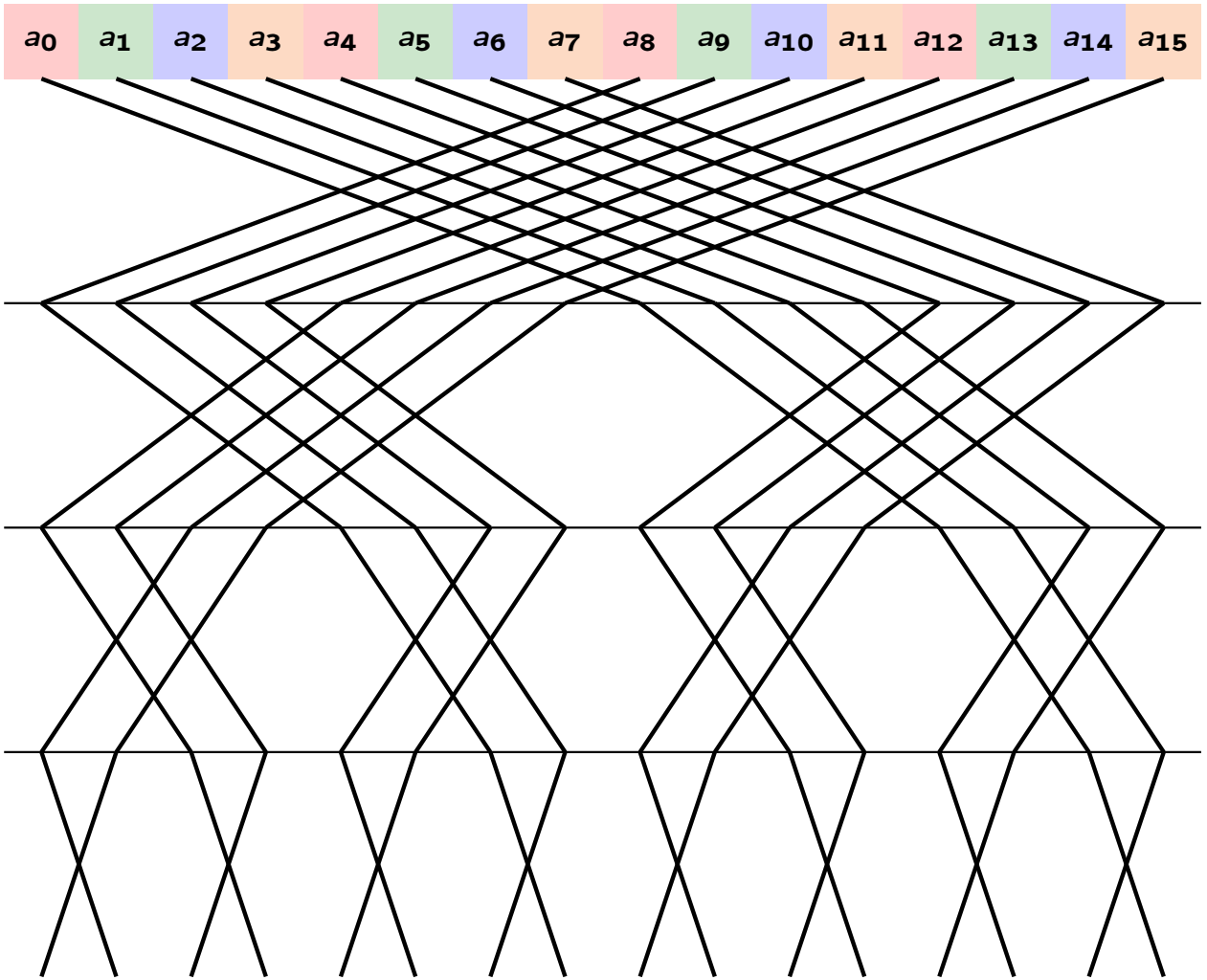
In $\mathbb{C}$, computing an FFT in $\{1, -1, i, -i\}$ is quite easy. But less obvious for superior orders...

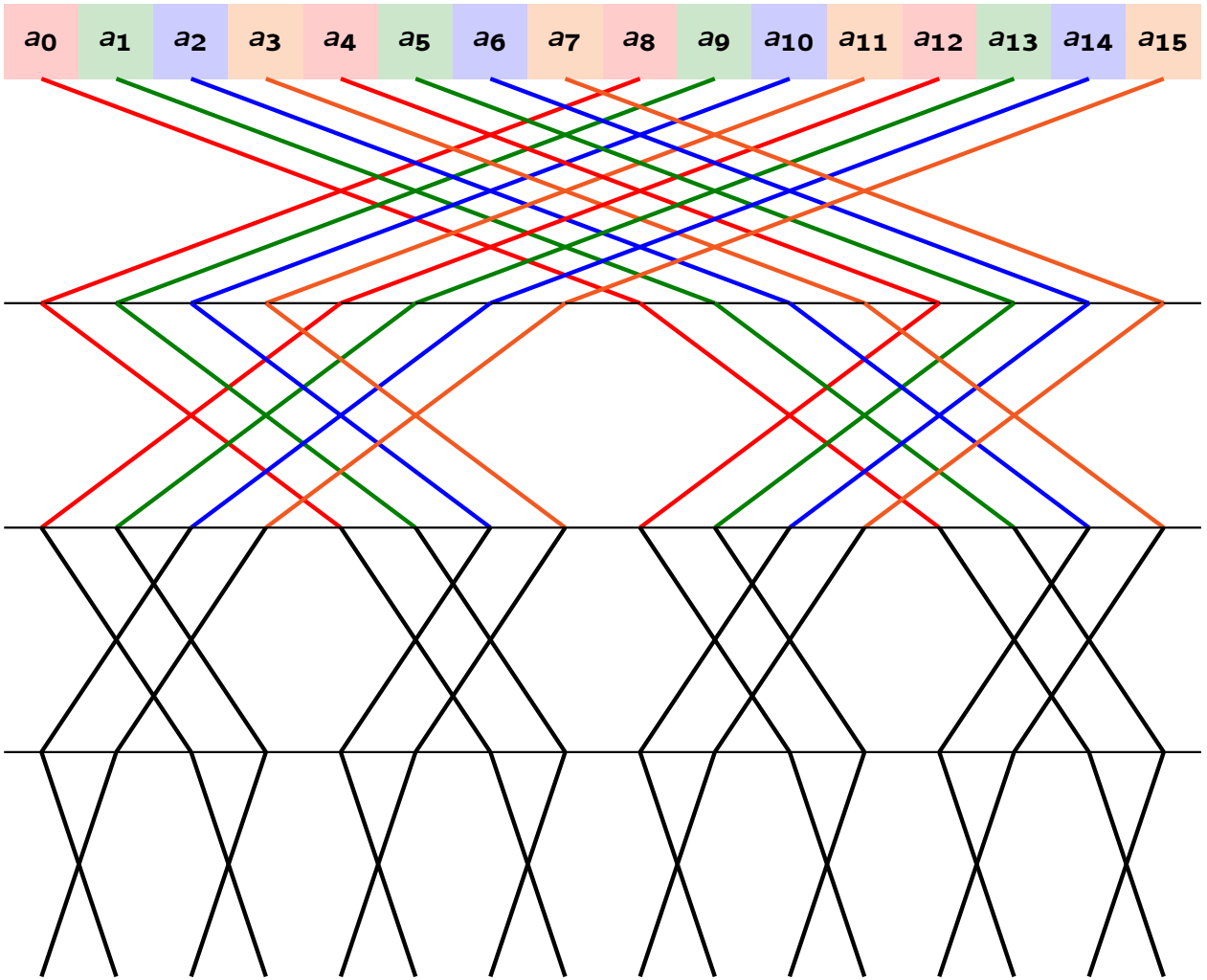
Here you can see the butterfly graph for 16-point transform.



Below, the matrix representation:

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 & a_7 \\ a_8 & a_9 & a_{10} & a_{11} \\ a_{12} & a_{13} & a_{14} & a_{15} \end{pmatrix}$$

We start by inner transforms.



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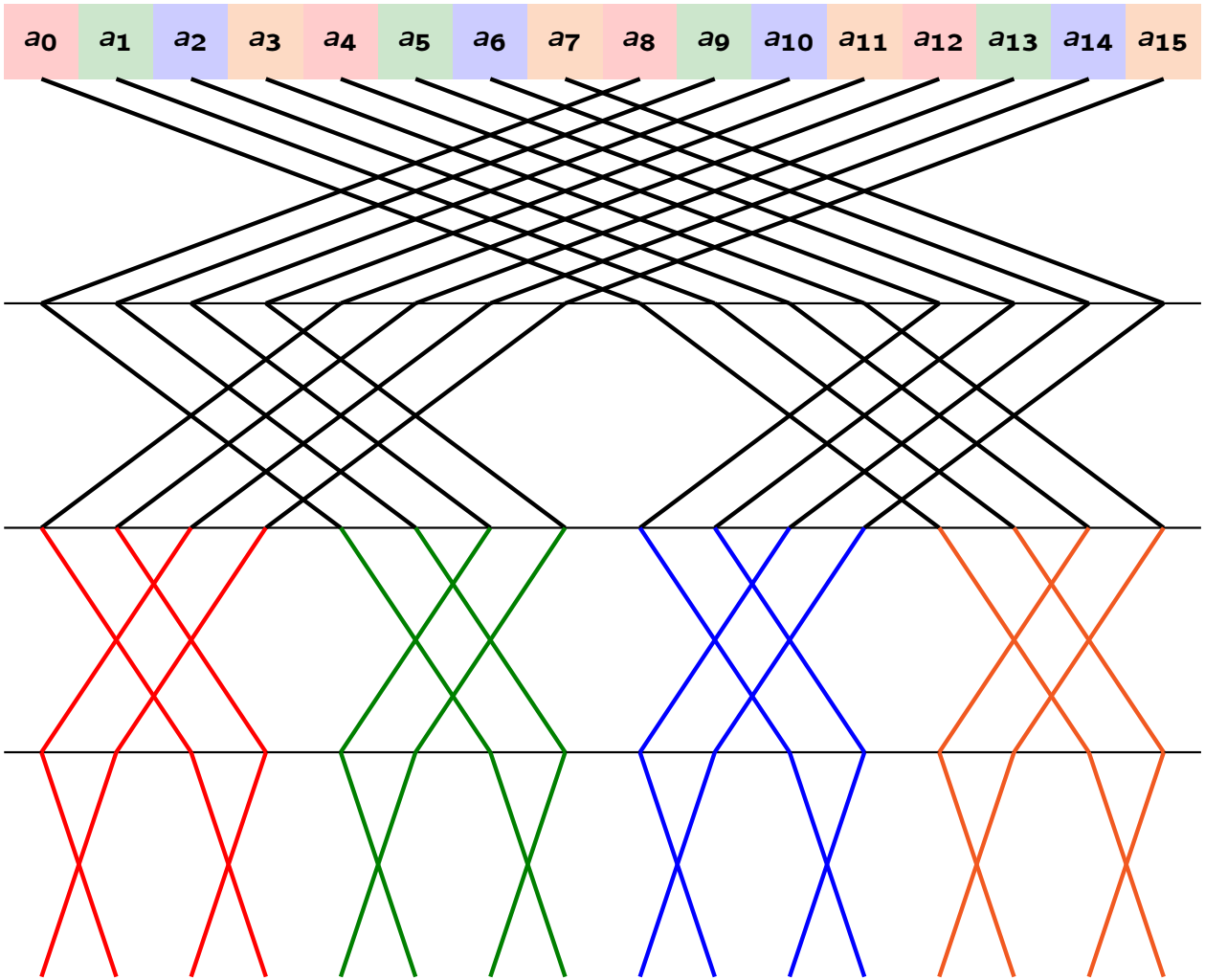
Then the outer ones.



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- There exists a $2n$-th root of unity ρ such that $\rho^{n/P} = x$ (Lagrange interpolation)
- The computation of $2n$-points FFT is factored into the computation of $\log_{2P} 2n$ times n/P $2P$-points FFT
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- q is chosen such that $q = b^P + 1$. There exists b such that $b < P \cdot (\log P)^{1+\epsilon}$ for any $\epsilon > 0$. Thus, $\log_2 q \approx P \log P$.
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	Naive Way	New way
x	x	$X(b) = x_0 + x_1 \cdot b + x_2 \cdot b^2 \cdots x_{P-1} \cdot b^{P-1}$
y	y	$Y(b) = y_0 + y_1 \cdot b + y_2 \cdot b^2 \cdots y_{P-1} \cdot b^{P-1}$
x $*$ y	$z = x \cdot y$ and $x * y = z \mod q$	$Z = X \cdot Y \mod (X^P + 1)$ and $x * y = Z(b)$

Using Fürer's algorithm, we got:

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Using the last trick, we get the following data:

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$\mathbb{Z}[x]/(q^c, x^P+1)\mathbb{Z}$	$O(N/\log^2 N)$	it depends	$O(\log^2 N)$	$N \log N\, 2^{O(\log^* N)}$
$\mathbb{Z}/(b^P+1)\mathbb{Z}$	$O(N/(\log N \log\log N))$	it depends	$O(\log N \log\log N)$	$N \log N\, 2^{O(\log^* N)}$

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An efficient implementation has to be done.

Some limitations: efficient multiplication of two polynomials modulo $X^P + 1$ (bilinear rank), strategy for choosing a good prime, an algorithm for the decompositions...