Fast computation of the Nth term of an algebraic series in positive characteristic

Philippe Dumas

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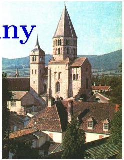

Joint work with
Alin Bostan and Gilles Christol

Ubiquity of algebraic functions (combinatorics, number theory)

Confluence of several domains:

- functional equations
- automatic sequences
- complexity theory

> *One of the most difficult questions in modular computations is the complexity of computations mod p for a large prime p of coefficients in the expansion of an algebraic function.*

D. Chudnovsky & G. Chudnovsky, 1990

Computer Algebra in the Service of

Mathematical Physics and Number Theory

Input:

- field $\mathbb{K}$ of characteristic p
- $f(x) \in \mathbb{K}[[x]]$ solution of $E(x, f(x)) = 0$ with $E(x, y) \in \mathbb{K}[x, y]$, $f(0) = 0$
- $N \in \mathbb{N}_{\geq 0}$

Output:

- the Nth coefficient from the series $f(x)$

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Output:

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Main result:

- arithmetic complexity linear in $\log N$ and almost linear in p

Method	char. 0	char. p
Undetermined coefficients	$O(N^d)$	✓
Fixed point iteration	$\widetilde{O}(N^2)$	✓
Newton iteration	$\widetilde{O}(N)$	✓
Linear recurrence	$O(N)$	✓*

*with p-adic computations

Kung + Traub, 1976
All algebraic functions can be computed fast

FFT used all along: multiplication cost for series at order N is $\widetilde{O}(N)$

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* with p-adic computations

not appropriate
for one coefficient

FFT used all along: multiplication cost for series at order N is $\widetilde{O}(N)$

Nth coefficient via shift operator

$$f(x) = \sum_{n \geq 0} f_n x^n$$

shift

$$S f_n = f_{n+1}$$

$$S f(x) = \frac{f(x) - f(0)}{x}$$

vector

operators

$$f(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \cdots$$

$$S f(x) = f_1 + f_2 x + f_3 x^2 + f_4 x^3 + \cdots$$

$$S^2 f(x) = f_2 + f_3 x + f_4 x^2 + f_5 x^3 + \cdots$$

$$\vdots$$

$$S^N f(x) = f_N + f_{N+1} x + f_{N+2} x^2 + \cdots$$

linear form

$$S^N f(0) = f_N$$

$$f(x) = \frac{a(x)}{b(x)} \qquad b(0) = 1$$

$$f(x) = \frac{a(x)}{b(x)} \qquad b(0) = 1$$

$$f(x) = \frac{1 + x^3}{1 - x - 2x^2 - 3x^3 - 4x^4 - 5x^5} = 1 + x + 3x^2 + 9x^3 + \cdots$$

$$Sf(x) = \frac{1 + 2x + 4x^2 + 4x^3 + 5x^4}{1 - x - 2x^2 - 3x^3 - 4x^4 - 5x^5} \qquad Sf(0) = 1$$

$$S^2 f(x) = \frac{3 + 6x + 7x^2 + 9x^3 + 5x^4}{1 - x - 2x^2 - 3x^3 - 4x^4 - 5x^5} \qquad S^2 f(0) = 3$$

$$S^3 f(x) = \frac{9 + 13x + 18x^2 + 17x^3 + 15x^4}{1 - x - 2x^2 - 3x^3 - 4x^4 - 5x^5} \qquad S^3 f(0) = 9$$

$$\vdots$$

$$f(x) = \frac{1+x^3}{1-x-2x^2-3x^3-4x^4-5x^5} = 1 + x + 3x^2 + 9x^3 + \cdots$$

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$$S^2f(x) = \frac{3+6x+7x^2+9x^3+5x^4}{1-x-2x^2-3x^3-4x^4-5x^5} \qquad S^2f(0) = 3$$

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$$\vdots$$

pseudo-shift operator T $\qquad$ $Sb^{-1} = b^{-1}T$

$\mathbb{K}[x]_{<d_x}$ stable by T $\qquad$ finite dimension d_x

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Linear form

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 1 \\ 5 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

basis: $(1, x, x^2, x^3, x^4)$

Action

$$\frac{1}{b} T1 = S\frac{1}{b} = \frac{\frac{1}{b} - \frac{1}{b(0)}}{x}$$
$$= \frac{1 + 2x + 3x^2 + 4x^3 + 5x^4}{b}$$

Coordinates

$$a(x) = 1 + x^3$$

$$f_N = S^N f(0) \quad \Leftrightarrow \quad f_N = LA^N C$$

Binary powering

example: $N = 1234$

$$A^{1234} = A^2 \times A^{16} \times A^{64} \times A^{128} \times A^{1024}$$

arithmetic complexity $O(\log_2 N)$

Miller + Brown, 1966, *An algorithm for evaluation of remote terms in a linear recurrence sequence*

- shift operator to express f_N
- space of polynomials with pseudo-shift
- finite dimensional space $\to$ linear representation
- binary powering $\to$ good arithmetic complexity

- shift operator to express f_N
- space of polynomials with pseudo-shift
- finite dimensional space $\rightarrow$ linear representation
- binary powering $\rightarrow$ good arithmetic complexity

To come: mimicking this approach in the algebraic case

Method	char. 0	char. p
Baby steps – Giant steps	$\widetilde{O}(\sqrt{N}) + O(1)$	✓*
Divide and Conquer	✘	$O(\log_p N) + \widetilde{O}(p^d)$

*with p-adic computations

Baby steps – Giant steps after precomputing

algebraic equation $\longrightarrow$ differential equation $\longrightarrow$ linear recurrence

Chudnovsky + Chudnovsky, 1988
Approximations and complex multiplication according to Ramanujan

Method	char. 0	char. p
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Baby steps – Giant steps after precomputing

algebraic equation $\longrightarrow$ differential equation $\longrightarrow$ linear recurrence

Divide and Conquer after precomputing

algebraic equation $\longrightarrow$ Mahler equation $\longrightarrow$ DAC recurrence

Christol + Kamae
+ Mendès France + Rauzy, 1980,
Suites algébriques, automates et substitutions

Algebraic equation $\quad y = 2x + 5xy + 4xy^2 + xy^3$

$p = 3$

$$
\begin{array}{c|cccc}
 & y & y^3 & y^9 & y^{27} \\
\hline
\begin{array}{c} 1 \\ y \\ y^2 \end{array} &
\multicolumn{4}{c}{\left[\begin{array}{cccc}
0 & 1 & \frac{1+2x^2+x^3}{x^3} & \frac{1+2x^2+x^3+2x^5+x^6+x^8+x^9+2x^{11}+x^{12}}{x^{12}} \\
1 & \frac{1+x}{x} & \frac{1+x+2x^2+x^3+x^4}{x^4} & \frac{1+x+2x^2+x^3+x^4+2x^5+2x^6+2x^7+x^8+x^9+x^{10}+2x^{11}+x^{12}+x^{13}}{x^{13}} \\
0 & 2 & 2\,\frac{1+x+2x^2+x^3}{x^3} & \frac{2+2x+x^2+2x^3+2x^4+x^5+x^6+x^7+2x^8+2x^9+2x^{10}+x^{11}+2x^{12}}{x^{12}}
\end{array}\right]}
\end{array}
$$

Mahler equation $\qquad c_0(x) = \left(2x^2 + 2x^3 + x^4\right)$

$$
\begin{aligned}
&c_0(x)f(x) \\
&\qquad + \left(1 + x^2 + 2x^3 + 2x^4 + x^5 + 2x^6\right) f(x^3) \\
&\qquad\qquad + \left(2 + 2x^3 + 2x^5 + x^6 + 2x^9\right) f(x^9) + x^9 f(x^{27}) = 0
\end{aligned}
$$

Frobenius $f(x)^p = f(x^p)$ $\qquad (a+b)^p = a^p + b^p$

Algebraic equation $y = 2x + 5xy + 4xy^2 + xy^3$

Mahler equation $c_0(x) = \left(2x^2 + 2x^3 + x^4\right)$

$$
\begin{aligned}
c_0(x)f(x) & \\
& + \left(1 + x^2 + 2x^3 + 2x^4 + x^5 + 2x^6\right) f(x^3) \\
& \quad + \left(2 + 2x^3 + 2x^5 + x^6 + 2x^9\right) f(x^9) + x^9 f(x^{27}) = 0
\end{aligned}
$$

Divide-and-conquer recurrence

$$
\begin{aligned}
2f_{n-2} + 2f_{n-3} + f_{n-4} & \\
& + f_{\frac{n}{3}} + f_{\frac{n-2}{3}} + 2f_{\frac{n-3}{3}} + 2f_{\frac{n-4}{3}} + f_{\frac{n-5}{3}} + 2f_{\frac{n-6}{3}} \\
& \quad + 2f_{\frac{n}{9}} + 2f_{\frac{n-3}{9}} + 2f_{\frac{n-5}{9}} + f_{\frac{n-6}{9}} + 2f_{\frac{n-9}{9}} + f_{\frac{n-9}{27}} = 0
\end{aligned}
$$

$f_x = 0$ if $x \notin \mathbb{N}_{\geq 0}$

Divide-and-conquer recurrence $p = 3$

$$2f_{n-2} + 2f_{n-3} + f_{n-4} + f_{\frac{n}{3}} + f_{\frac{n-2}{3}} + 2f_{\frac{n-3}{3}} + 2f_{\frac{n-4}{3}} + f_{\frac{n-5}{3}} + 2f_{\frac{n-6}{3}} + 2f_{\frac{n}{9}} + 2f_{\frac{n-3}{9}} + 2f_{\frac{n-5}{9}} + f_{\frac{n-6}{9}} + 2f_{\frac{n-9}{9}} + f_{\frac{n-9}{27}} = 0$$

$N = 100$

100

Divide-and-conquer recurrence $p = 3$

$$
\begin{aligned}
&2f_{n-2} + 2f_{n-3} + f_{n-4} \\
&\qquad + f_{\frac{n}{3}} + f_{\frac{n-2}{3}} + 2f_{\frac{n-3}{3}} + 2f_{\frac{n-4}{3}} + f_{\frac{n-5}{3}} + 2f_{\frac{n-6}{3}} \\
&\qquad\quad + 2f_{\frac{n}{9}} + 2f_{\frac{n-3}{9}} + 2f_{\frac{n-5}{9}} + f_{\frac{n-6}{9}} + 2f_{\frac{n-9}{9}} + f_{\frac{n-9}{27}} = 0
\end{aligned}
$$

$N = 100$

100 99 98

34 33 32

11

Divide-and-conquer recurrence $p = 3$

$$
\begin{aligned}
&2f_{n-2} + 2f_{n-3} + f_{n-4} \\
&\qquad + f_{\frac{n}{3}} + f_{\frac{n-2}{3}} + 2f_{\frac{n-3}{3}} + 2f_{\frac{n-4}{3}} + f_{\frac{n-5}{3}} + 2f_{\frac{n-6}{3}} \\
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\end{aligned}
$$

$N = 100$

100 99 98 97 96

34 33 32 31 30

12 11 10 9

4 3 1

Divide-and-conquer recurrence $p = 3$

$$2f_{n-2} + 2f_{n-3} + f_{n-4} + f_{\frac{n}{3}} + f_{\frac{n-2}{3}} + 2f_{\frac{n-3}{3}} + 2f_{\frac{n-4}{3}} + f_{\frac{n-5}{3}} + 2f_{\frac{n-6}{3}} + 2f_{\frac{n}{9}} + 2f_{\frac{n-3}{9}} + 2f_{\frac{n-5}{9}} + f_{\frac{n-6}{9}} + 2f_{\frac{n-9}{9}} + f_{\frac{n-9}{27}} = 0$$

$N = 100$

100 99 98 97 96 95 94 ...

34 33 32 31 30 29 28 ...

12 11 10 9 8 7 ...

4 3 2 1 0

arithmetic complexity $O(N)$ for f_N

Change of unknowns $f(x) = c_0(x)g(x)$ yields

$$
\begin{aligned}
g(x) = {} & (x^2 + x^3 + x^7 + x^8 + x^9 + x^{10})g(x^3) \\
& + \big(2x^{14} + 2x^{15} + x^{16} + 2x^{19} + 2x^{20} + x^{21} + x^{22} + 2x^{23} \\
& + 2x^{26} + x^{28} + 2x^{30} + x^{31} + 2x^{32} + x^{33} + 2x^{35} + 2x^{36} + x^{37}\big)\, g(x^9) \\
& + \big(x^{59} + x^{60} + 2x^{61} + 2x^{62} + 2x^{63} + x^{64} + 2x^{65} + 2x^{66} + x^{67} + 2x^{71} \\
& + 2x^{72} + x^{73} + x^{74} + x^{75} + 2x^{76} + x^{77} + x^{78} + 2x^{79} + x^{83} + x^{84} \\
& + 2x^{85} + x^{89} + x^{90} + 2x^{91} + 2x^{92} + 2x^{93} + x^{94} + 2x^{95} + 2x^{96} \\
& + x^{97} + 2x^{101} + 2x^{102} + x^{103} + x^{104} + x^{105} + 2x^{106} + x^{107} \\
& + x^{108} + 2x^{109}\big)\, g(x^{27})
\end{aligned}
$$

Change of unknowns $f_n = 2g_{n-2} + 2g_{n-3} + g_{n-4}$

$$
\begin{aligned}
g_n = {} & g_{\frac{n-2}{3}} + g_{\frac{n-3}{3}} + g_{\frac{n-7}{3}} + g_{\frac{n-8}{3}} + g_{\frac{n-9}{3}} + g_{\frac{n-10}{3}} \\
& + 2g_{\frac{n-14}{9}} + 2g_{\frac{n-15}{9}} + g_{\frac{n-16}{9}} + 2g_{\frac{n-19}{9}} + 2g_{\frac{n-20}{9}} + \cdots + g_{\frac{n-37}{9}} \\
& + g_{\frac{n-59}{27}} + g_{\frac{n-60}{27}} + \cdots + 2g_{\frac{n-109}{27}}
\end{aligned}
$$

$N = 100$

98 97 96

Change of unknowns $f_n = 2g_{n-2} + 2g_{n-3} + g_{n-4}$

$$\begin{aligned} g_n = {} & g_{\frac{n-2}{3}} + g_{\frac{n-3}{3}} + g_{\frac{n-7}{3}} + g_{\frac{n-8}{3}} + g_{\frac{n-9}{3}} + g_{\frac{n-10}{3}} \\ & + 2g_{\frac{n-14}{9}} + 2g_{\frac{n-15}{9}} + g_{\frac{n-16}{9}} + 2g_{\frac{n-19}{9}} + 2g_{\frac{n-20}{9}} + \cdots + g_{\frac{n-37}{9}} \\ & + g_{\frac{n-59}{27}} + g_{\frac{n-60}{27}} + \cdots + 2g_{\frac{n-109}{27}} \end{aligned}$$

$N = 100$

98 97 96

32 31 30 29

9 8 7

1 0

Change of unknowns $f_n = 2g_{n-2} + 2g_{n-3} + g_{n-4}$

$$\begin{aligned} g_n = {} & g_{\frac{n-2}{3}} + g_{\frac{n-3}{3}} + g_{\frac{n-7}{3}} + g_{\frac{n-8}{3}} + g_{\frac{n-9}{3}} + g_{\frac{n-10}{3}} \\ & + 2g_{\frac{n-14}{9}} + 2g_{\frac{n-15}{9}} + g_{\frac{n-16}{9}} + 2g_{\frac{n-19}{9}} + 2g_{\frac{n-20}{9}} + \cdots + g_{\frac{n-37}{9}} \\ & + g_{\frac{n-59}{27}} + g_{\frac{n-60}{27}} + \cdots + 2g_{\frac{n-109}{27}} \end{aligned}$$

$N = 100$

98 97 96

32 31 30 29

10 9 8 7

2 1 0

arithmetic complexity $O(\log_p N)$ for f_N

Introduction

Special case of rational series

Algorithms with low complexity

Sections and diagonal

Partial powering

Section operators

$$S_r \sum_{n\geq 0} u_n x^n = \sum_{k\geq 0} u_{pk+r} x^k, \qquad 0 \leq r < p$$

Section operators

$$S_r \sum_{n\geq 0} u_n x^n = \sum_{k\geq 0} u_{pk+r} x^k, \qquad 0 \leq r < p$$

$$\begin{aligned} f(x) &= f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 + \cdots \\ S_0 f(x) &= f_0 + f_2 x + f_4 x^2 + f_6 x^3 + f_8 x^4 + \cdots \\ S_1 f(x) &= f_1 + f_3 x + f_5 x^2 + f_7 x^3 + f_9 x^4 + \cdots \end{aligned} \qquad p = 2$$

linear operators $\quad S_r\big(\,g(x) \times h(x^p)\,\big) = \big(\,S_r g(x)\,\big) \times h(x)$

$$
\begin{aligned}
100000 &= 5 \times 16807 + 6 \times 2401 + 4 \times 343 + 3 \times 49 + 5 \times 7 + 5 \times 1 \\
&= (5, 6, 4, 3, 5, 5)_7
\end{aligned}
$$

$$
\begin{aligned}
f(x) &= f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 + f_5 x^5 + \cdots \\
S_5 f(x) &= f_5 + f_{12} x + f_{19} x^2 + f_{26} x^3 + f_{33} x^4 + f_{40} x^5 + \cdots \\
S_5 S_5 f(x) &= f_{40} + f_{89} x + f_{138} x^2 + f_{187} x^3 + f_{236} x^4 + f_{285} x^5 + \cdots \\
S_3 S_5 S_5 f(x) &= f_{187} + f_{530} x + f_{873} x^2 + f_{1216} x^3 + f_{1559} x^4 + f_{1902} x^5 + \cdots \\
S_4 S_3 S_5 S_5 f(x) &= f_{1559} + f_{3960} x + f_{6361} x^2 + f_{8762} x^3 + f_{11163} x^4 + \cdots \\
S_6 S_4 S_3 S_5 S_5 f(x) &= f_{15965} + f_{32772} x + f_{49579} x^2 + f_{66386} x^3 + f_{83193} x^4 + \cdots \\
S_5 S_6 S_4 S_3 S_5 S_5 f(x) &= f_{100000} + f_{217649} x + f_{335298} x^2 + f_{452947} x^3 + \cdots \\
S_5 S_6 S_4 S_3 S_5 S_5 f(0) &= f_{100000}
\end{aligned}
$$

Nth coefficient via section operators

vector

$$
\begin{aligned}
f(x) &= f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 + f_5 x^5 + \cdots \\
S_5 f(x) &= f_5 + f_{12} x + f_{19} x^2 + f_{26} x^3 + f_{33} x^4 + f_{40} x^5 + \cdots \\
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S_5 S_6 S_4 S_3 S_5 S_5 f(0) &= f_{100000}
\end{aligned}
$$

operators

linear form

$$E(x, f(x)) = 0 \qquad E(x,y) \in \mathbb{K}[x,y]$$

$$\Updownarrow$$

$$f(x) = \mathcal{D}F(x,y) \qquad F(x,y) \in \mathbb{K}(x,y)$$

Furstenberg, 1967,
Algebraic functions over finite fields

Christol, 1974,
Éléments algébriques

$$E(x,f(x))=0 \qquad E(x,y)\in\mathbb{K}[x,y]$$

$$\Updownarrow$$

$$f(x)=\mathcal{D}F(x,y) \qquad F(x,y)\in\mathbb{K}(x,y)$$

$E_y(0,0)\neq 0$

$$F(x,y)=y^2\,\frac{E_y(xy,y)}{E(xy,y)}$$

$$E(x,y)=2x+(5x-1)\,y+4xy^2+xy^3$$

$$F(x,y)=\frac{y(1-5xy-8xy^2-3xy^3)}{1-2x-5xy-4xy^2-xy^3}$$

Furstenberg, 1967, *Algebraic functions over finite fields*

Christol, 1974, *Éléments algébriques*

$$f(x) = 2x + 3x^2 + 3x^3 + x^4 + 6x^5 + 5x^6 + 6x^7 + 3x^8 + x^9 + 4x^{10} + x^{14} + 5x^{15} + \cdots$$

$$p = 7$$

$$F(x, y) =$$

$$\begin{array}{ccccccccccc}
y & + 2xy & + 4x^2y & + x^3y & + 2x^4y & + 4x^5y & + x^6y & + 2x^7y & + 4x^8y & + x^9y & + 2x^{10}y \\
 & & + 3x^2y^2 & + 5x^3y^2 & + x^4y^2 & + 5x^5y^2 & + 2x^6y^2 & + 2x^7y^2 & & + 6x^9y^2 & + 3x^{10}y^2 \\
 & + 3xy^3 & & + 3x^3y^3 & & + 6x^5y^3 & + 4x^6y^3 & + x^7y^3 & + 6x^8y^3 & & + 6x^{10}y^3 \\
 & + 5xy^4 & + 6x^2y^4 & & + x^4y^4 & + x^5y^4 & & + x^7y^4 & + 3x^8y^4 & + 5x^9y^4 & \\
 & & + 2x^2y^5 & + 2x^3y^5 & & + 6x^5y^5 & + 4x^6y^5 & + 5x^7y^5 & & + 4x^9y^5 & + 4x^{10}y^5 \\
 & & + 2x^2y^6 & + 3x^3y^6 & + 5x^4y^6 & & + 5x^6y^6 & + 6x^7y^6 & & + 4x^9y^6 & + 6x^{10}y^6 \\
 & & + 5x^2y^7 & + 6x^3y^7 & & + 5x^5y^7 & & + 6x^7y^7 & & + 3x^9y^7 & + 5x^{10}y^7 \\
 & & & & + 2x^4y^8 & + 3x^5y^8 & + 2x^6y^8 & & + 3x^8y^8 & + 6x^9y^8 & + 5x^{10}y^8 \\
 & & & + x^3y^9 & + x^4y^9 & + 3x^5y^9 & + 6x^6y^9 & + 6x^7y^9 & & + x^9y^9 & + 6x^{10}y^9 \\
 & & & + 5x^3y^{10} & & + 6x^5y^{10} & + 2x^6y^{10} & & + x^8y^{10} & & + 4x^{10}y^{10} \\
 & & & & + x^4y^{11} & + x^5y^{11} & + 5x^6y^{11} & + 4x^7y^{11} & + 4x^8y^{11} & + 2x^9y^{11} & \\
 & & & & & + 6x^5y^{12} & + 2x^6y^{12} & + x^7y^{12} & & + 3x^9y^{12} & + 3x^{10}y^{12} \\
 & & & & + 5x^4y^{13} & & + 5x^6y^{13} & & & + 3x^9y^{13} & + x^{10}y^{13} \\
 & & & & & + 5x^5y^{14} & & + 3x^7y^{14} & & + 4x^9y^{14} & + 2x^{10}y^{14} \\
 & & & & & + 6x^5y^{15} & + x^6y^{15} & + 3x^7y^{15} & + x^8y^{15} & + 2x^9y^{15} & + 4x^{10}y^{15} \\
 & & & & & + 5x^5y^{16} & + 4x^6y^{16} & + 4x^7y^{16} & & + 5x^9y^{16} & + 4x^{10}y^{16} \\
 & & & & & & + \cdots & & & &
\end{array}$$

$$S_r f = S_r \mathcal{D} F = \mathcal{D} S_{r,r} F$$

$$F = \frac{a}{b} = b^{-1} a$$

$$b(0,0) = 1$$

$$S_r f = \mathcal{D} S_{r,r} b^{-1} a$$

$$= \mathcal{D} S_{r,r} b^{-p} b^{p-1} a = \mathcal{D} b^{-1} S_{r,r} b^{p-1} a$$

$$S_r f = \mathcal{D} b^{-1} T_r a$$

$$T_r = S_{r,r} b^{p-1}$$

T_r pseudo-section operator

$$S_{r,r} b^{-p} = b^{-1} T_r$$

f is algebraic $\Leftrightarrow$ f is rational with respect to the radix p

Christol + Kamae + Mendès France + Rauzy, 1980, Suites algébriques, automates et substitutions

Allouche + Shallit, 1992 The ring of k-regular sequences

f is algebraic	$\Leftrightarrow$	f is rational with respect to the radix p

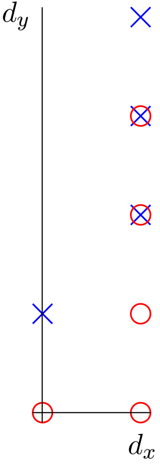
$$f = \mathcal{D}\frac{a}{b}$$

f is algebraic	$\Leftrightarrow$	a generates a finite dimensional vector space under the action of the pseudo-section operators T_r.

Algebraic equation $\quad y = 2x + 5xy + 4xy^2 + xy^3 \qquad\qquad f = \mathcal{D}\frac{a}{b}$

$a = y + 2xy^2 + 6xy^3 + 4xy^4$

$b = 1 + 5x + 2xy + 3xy^2 + 6xy^3$

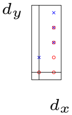


$$d_x = \max(\deg_x a, \deg_x b)$$

$$d_y = \max(\deg_y a, \deg_y b)$$

$$a = y + 2xy^2 + 6xy^3 + 4xy^4$$

$$b = 1 + 5x + 2xy + 3xy^2 + 6xy^3$$

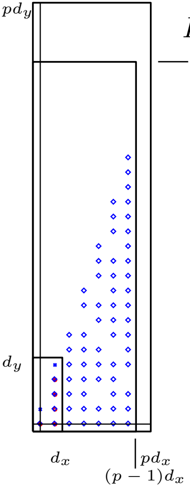


$a = y + 2xy^2 + 6xy^3 + 4xy^4$ ×

$b = 1 + 5x + 2xy + 3xy^2 + 6xy^3$ ○

$B = b^{p-1}$ ◇

$p = 7$

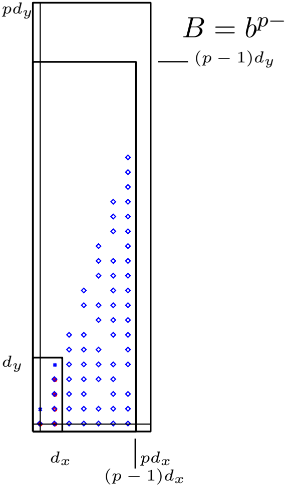


$T_r = S_{r,r}B$

$a = y + 2xy^2 + 6xy^3 + 4xy^4$ ×

$b = 1 + 5x + 2xy + 3xy^2 + 6xy^3$ ○

$B = b^{p-1}$ ◇



$p = 7$

$T_r = S_{r,r}B$

$\mathbb{K}[x,y]_{\le d_x, \le d_y}$ stable by T_r, $0 \le r < p$

finite dimension $(1+d_x)(1+d_y)$

Rational series wrt p and linear representation

$$\begin{aligned}100000 &= 5\times 16807 + 6\times 2401 + 4\times 343 + 3\times 49 + 5\times 7 + 5\times 1\\ &= (5,6,4,3,5,5)_7\end{aligned}$$

vector

$$\begin{aligned}
f(x) &= f_0 + f_1x + f_2x^2 + f_3x^3 + f_4x^4 + f_5x^5 + \cdots\\
S_5f(x) &= f_5 + f_{12}x + f_{19}x^2 + f_{26}x^3 + f_{33}x^4 + f_{40}x^5 + \cdots\\
S_5S_5f(x) &= f_{40} + f_{89}x + f_{138}x^2 + f_{187}x^3 + f_{236}x^4 + f_{285}x^5 + \cdots\\
S_3S_5S_5f(x) &= f_{187} + f_{530}x + f_{873}x^2 + f_{1216}x^3 + f_{1559}x^4 + f_{1902}x^5 + \cdots\\
S_4S_3S_5S_5f(x) &= f_{1559} + f_{3960}x + f_{6361}x^2 + f_{8762}x^3 + f_{11163}x^4 + \cdots\\
S_6S_4S_3S_5S_5f(x) &= f_{15965} + f_{32772}x + f_{49579}x^2 + f_{66386}x^3 + f_{83193}x^4 + \cdots\\
S_5S_6S_4S_3S_5S_5f(x) &= f_{100000} + f_{217649}x + f_{335298}x^2 + f_{452947}x^3 + \cdots\\
S_5S_6S_4S_3S_5S_5f(0) &= f_{100000}
\end{aligned}$$

operators

linear form

$$\begin{aligned} 100000 &= 5 \times 16807 + 6 \times 2401 + 4 \times 343 + 3 \times 49 + 5 \times 7 + 5 \times 1 \\ &= (5, 6, 4, 3, 5, 5)_7 \end{aligned}$$

vector

$$\begin{aligned} a(x,y) &= y + 2xy^2 + 6xy^3 + 4xy^4 \\ T_5 a(x,y) &= 6 + 6y \\ T_5 T_5 a(x,y) &= 5 + 5y \\ T_3 T_5 T_5 a(x,y) &= 5 + 5y \\ T_4 T_3 T_5 T_5 a(x,y) &= 2 + 2y \\ T_6 T_4 T_3 T_5 T_5 a(x,y) &= 2 + 2y \\ T_5 T_6 T_4 T_3 T_5 T_5 a(x,y) &= 4 + 4y \\ T_5 T_6 T_4 T_3 T_5 T_5 a(0,0) &= f_{100000} = 4 \end{aligned}$$

operators

linear form

finite dimension $(1 + d_x)(1 + d_y)$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Linear form

$$A_0, \quad A_1, \quad \dots \quad A_6$$

$$C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 2 \\ 0 & 6 \\ 0 & 4 \end{bmatrix}$$

Action

basis: $\begin{matrix} 1 & x \\ y & xy \\ y^2 & xy^2 \\ y^3 & xy^3 \\ y^4 & xy^4 \end{matrix}$

Coordinates

$$a(x, y) = y + 2xy^2 + 6xy^3 + 4xy^4$$

finite dimension $(1 + d_x)(1 + d_y)$

$N = (N_{\ell-1} \ldots N_1 N_0)_p$

$$f_N = S_{N_{\ell-1}} \cdots S_{N_0} f(0)$$

$$\Updownarrow$$

$$f_N = T_{N_{\ell-1}} \cdots T_{N_0} a(0,0)$$

$$\Updownarrow$$

$$f_N = L A_{N_{\ell-1}} \cdots A_{N_0} C$$

arithmetic complexity $O(\log_p N)$

Allouche + Shallit, 1992
The ring of k-regular sequences

All information is in $B = b^{p-1}$.

$$\text{matrix } A_r\colon\ x^n y^m \xrightarrow[\text{translation}]{} x^n y^m B \xrightarrow[\text{selection}]{} S_{r,r} x^n y^m B$$

No computation, except raising b to the power $p-1$

cost: $\widetilde{O}(p^2)$ (binary powering, Kronecker substitution, FFT)

precomputation	computation
$B = b^{p-1}$	$f_N = LA_{N_{\ell-1}} \cdots A_{N_0} C$
$\widetilde{O}(p^2)$	$O(\log_p N)$

Theorem [New]: The Nth coefficient can be computed in time

$$\widetilde{O}(p^2) + O(\log_p N).$$

Task: Computation of A_0, A_1, ..., A_{p-1}

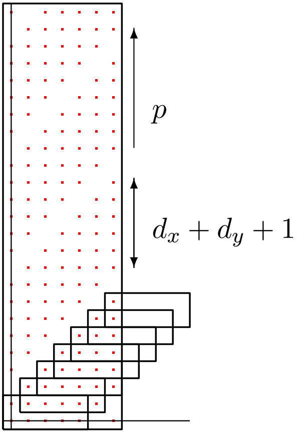
row index $i = (k, \ell)$
column index $j = (n, m)$

$$B = \sum_{\alpha,\beta} c_{\alpha,\beta} x^\alpha y^\beta$$

$$x^n y^m \xrightarrow[\text{translation}]{} x^n y^m B \xrightarrow[\text{selection}]{} S_{r,r} x^n y^m B$$

$$x^n y^m \longrightarrow \sum_{\alpha,\beta} c_{\alpha,\beta} x^{n+\alpha} y^{m+\beta} \longrightarrow \sum_{\substack{\alpha,\beta \\ (C)}} c_{\alpha,\beta} x^k y^\ell$$

$$(C) \begin{cases} n + \alpha = pk + r \\ m + \beta = p\ell + r \end{cases} \implies \beta - \alpha = p(\ell - k) + n - m$$

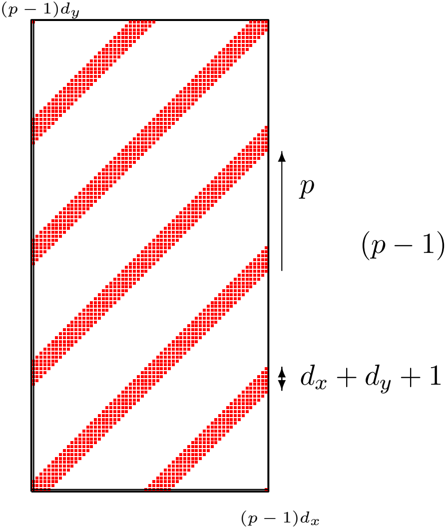


$p = 7$

$d_x = 1$

$d_y = 4$

Only a small part of $B = b^{p-1}$ is enough



$p = 31$

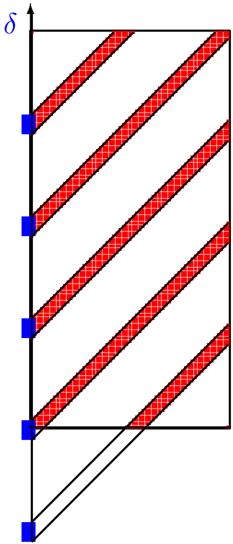
$d_x = 2$

$d_y = 4$

$$(p-1)d_x \times (p-1)d_y \times \frac{d_x + d_y + 1}{p} = O(p)$$

$$B(x/t,t) = \sum_{\alpha,\beta} c_{\alpha,\beta} x^{\alpha} t^{\beta-\alpha} = \sum_{\delta} \pi_{\delta}(x) t^{\delta}$$

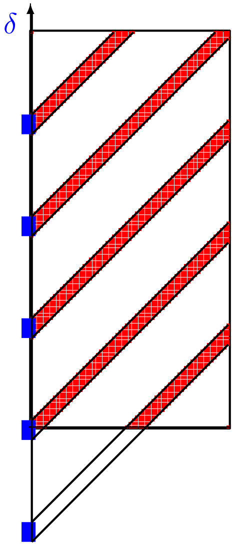
$\delta = \beta - \alpha = p(\ell - k) + n - m$



$$B(x/t,t) = \sum_{\alpha,\beta} c_{\alpha,\beta} x^\alpha t^{\beta-\alpha} = \sum_{\delta} \pi_\delta(x) t^\delta$$

$$\delta = \beta - \alpha = p(\ell - k) + n - m$$

$$B(x/t,t) = b(x/t,t)^{p-1} = \frac{b(x/t,t)^p}{b(x/t,t)} = \frac{b(x^p/t^p,t^p)}{b(x/t,t)}$$



$$B(x/t,t) = \sum_{\alpha,\beta} c_{\alpha,\beta} x^{\alpha} t^{\beta-\alpha} = \sum_{\delta} \pi_{\delta}(x) t^{\delta}$$

$$\delta = \beta - \alpha = p(\ell - k) + n - m$$

$$B(x/t,t) = b(x/t,t)^{p-1} = \frac{b(x/t,t)^p}{b(x/t,t)} = \frac{b(x^p/t^p,t^p)}{b(x/t,t)}$$

$$\frac{1}{b(x/t,t)} = \sum_u b_u^{(0)}(x) t^u \qquad b(x^p/t^p,t^p) = \textstyle\sum_v b_v^{(1)}(x^p) t^{pv}$$

rational series — for free

$$B(x/t,t) = \sum_{\alpha,\beta} c_{\alpha,\beta} x^{\alpha} t^{\beta-\alpha} = \sum_{\delta} \pi_{\delta}(x) t^{\delta}$$

$$B(x/t,t) = b(x/t,t)^{p-1} = \frac{b(x/t,t)^p}{b(x/t,t)} = \frac{b(x^p/t^p,t^p)}{b(x/t,t)}$$

$$\frac{1}{b(x/t,t)} = \sum_{u} b_u^{(0)}(x) t^u \qquad b(x^p/t^p,t^p) = \textstyle\sum_v b_v^{(1)}(x^p) t^{pv}$$

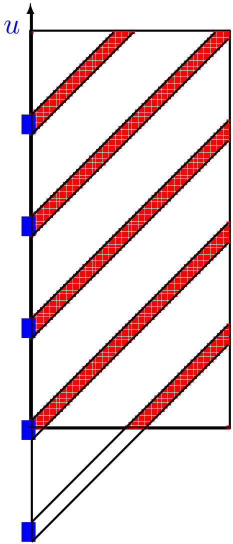
rational series — for free

$$\pi_{\delta}(x) = \sum_{u+pv=\delta} b_u^{(0)}(x) b_v^{(1)}(x^p) \qquad u + pv = \delta \Longrightarrow u \equiv \delta \mod p$$

same intervals

$$\frac{1}{b(x/t,t)} = \sum_u b_u^{(0)}(x) t^u \in \mathbb{K}(x)[[t]]$$

rational series



same intervals

$$\frac{1}{b(x/t,t)} = \sum_u b_u^{(0)}(x)t^u \in \mathbb{K}(x)[[t]]$$

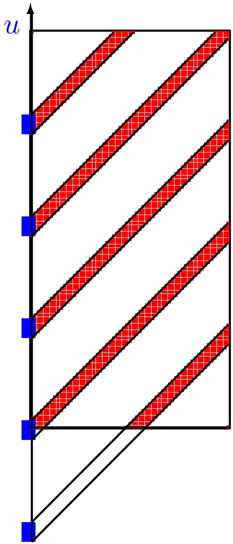
$$\frac{1}{b(\xi/t,t)} = \sum_u b_u^{(0)}(\xi)t^u \in \mathbb{K}[[t]], \qquad \xi \in \mathbb{K}$$

rational series with coefficients in $\mathbb{K}$

binary powering

about $d_x + d_y = O(1)$ large leaps of length p

$\longrightarrow$ cost $O(\log p)$



binary powering

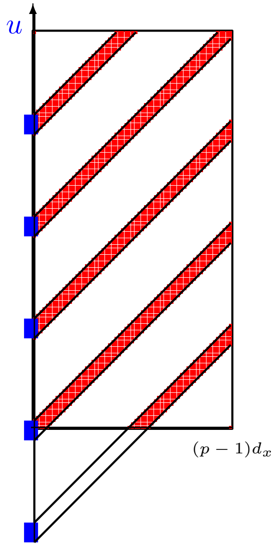
about $d_x + d_y = O(1)$ large leaps of length p

$\longrightarrow$ cost $O(\log p)$

interpolation

degree about $(p-1)d_x = O(p)$

$\longrightarrow$ cost $\widetilde{O}(p \log^2 p)$



binary powering

about $d_x + d_y$ large leaps of length p

$\longrightarrow$ cost $(d_x + d_y)\log p$

interpolation

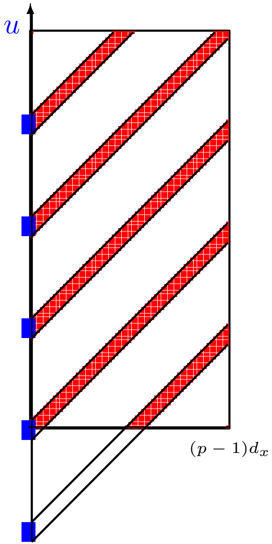
degree about pd_x

$\longrightarrow$ cost $\widetilde{O}(p\log^2 p)$

gathering

about $(d_x + d_y)^2 = O(1)$ indices

$\longrightarrow$ global cost $\widetilde{O}(p\log^2 p)$



Theorem [New and better]: The Nth coefficient can be computed in time

$$\widetilde{\boldsymbol{O}}(\boldsymbol{p}) + O(\log_p N).$$

Not quite obvious result following from the asymptotic expansions of the difference equation, shows that [...] requires only $O(p \cdot \log_p N)$ operations.

D. Chudnovsky & G. Chudnovsky, 1990

Computer Algebra in the Service of

Mathematical Physics and Number Theory

Theorem [New and better]: The Nth coefficient can be computed in time

$$\widetilde{\boldsymbol{O}}(\boldsymbol{p}) + O(\log_p N).$$

What is the minimal complexity of the computation of N-th coefficient of a power series expansion of an algebraic function mod p *for a fixed (large) prime p? It is reasonable to conjecture that [...] one needs at most $O(L(p)\log N)$ operations, where $L(p) = \exp(\sqrt{\log p \,\log\log p})$ at least for hyperelliptic algebraic functions.*

D. Chudnovsky & G. Chudnovsky, 1987

Computer assisted number theory with applications

Thank you for your attention!

$$+5x^{131}+2x^{130}+4x^{129}+x^{128}+2x^{127}+5x^{126}+3x^{125}+2x^{124}+4x^{123}+x^{122}+x^{121}+6x^{120}+6x^{119}$$
$$+2x^{118}+2x^{117}+5x^{116}+3x^{115}+6x^{114}+4x^{113}+2x^{112}+6x^{111}+4x^{110}+6x^{109}+5x^{108}+6x^{107}$$
$$+5x^{106}+6x^{105}+4x^{104}+3x^{103}+4x^{102}+x^{101}+2x^{100}+5x^{99}+3x^{98}+4x^{97}+5x^{71}+4x^{69}+2x^{68}+2x^{67}$$
$$+4x^{66}+5x^{65}+3x^{64}+x^{62}+2x^{57}+3x^{55}+3x^{54}+3x^{53}+6x^{52}+4x^{51}+5x^{50}+4x^{48}+2x^{46}+4x^{45}$$
$$+3x^{44}+x^{43}+6x^{42}+5x^{41}+2x^{40}\text{[illegible]}x^{39}\text{[illegible]}x^{38}+6x^{37}+3x^{36}+2x^{35}+x^{34}+4x^{33}\text{[illegible]}x^{32}+6x^{31}+5x^{30}$$
$$+3x^{29}+4x^{28}+x^{27}+3x^{26}+6x^{25}+5x^{24}+5x^{23}+5x^{22}+2x^{21}+4x^{20}+x^{18}+2x^{17}+5x^{16}+5x^{15}+3x^{14}$$
$$+4x^{13}+6x^{12}\text{[illegible]}x^{11}+4x^{10}+2x^{9}+x^{8}+2x^{7}+5x^{6}\text{[illegible]}+3x^{3}+4x^{2}+1\Big)f(x^{7})$$
$$+\Big(6x^{165}+5x^{164}+2x^{163}+2x^{162}+4x^{161}+3x^{160}+2x^{155}+3x^{153}+2x^{151}+4x^{150}+3x^{149}+6x^{147}$$
$$+x^{144}+2x^{143}+5x^{142}+4x^{141}+3x^{140}+6x^{139}+x^{137}+2x^{136}+5x^{135}+x^{134}+3x^{133}+5x^{132}+2x^{130}$$
$$+4x^{129}+3x^{128}+4x^{127}+6x^{126}+6x^{125}+6x^{123}+5x^{122}+2x^{121}+2x^{120}+4x^{119}+3x^{118}+5x^{116}$$
$$+3x^{115}+4x^{114}+4x^{113}+x^{112}+6x^{111}+4x^{106}+6x^{104}+4x^{102}+x^{101}+6x^{100}+5x^{98}+2x^{71}+3x^{69}$$
$$+x^{67}+2x^{66}+5x^{65}+5x^{64}+3x^{63}+4x^{62}+5x^{57}+4x^{55}+5x^{53}+3x^{52}+4x^{51}+x^{49}+5x^{46}+3x^{45}+4x^{44}$$
$$+6x^{43}+x^{42}+2x^{41}+5x^{39}+3x^{38}+4x^{37}+5x^{36}+x^{35}+4x^{34}+3x^{32}+6x^{31}+x^{30}+6x^{29}+2x^{28}+2x^{27}$$
$$+2x^{25}+4x^{24}+3x^{23}+6x^{21}+6x^{18}+5x^{17}+2x^{16}+2x^{15}+4x^{14}+3x^{13}+2x^{8}+3x^{6}+2x^{4}+4x^{3}+3x^{2}+6\Big)f(x^{49})$$
$$+\Big(x^{165}+2x^{164}+5x^{163}+5x^{162}+3x^{161}+4x^{160}+5x^{155}+4x^{153}+5x^{151}+3x^{150}+4x^{149}+x^{147}\Big)f(x^{343})=0$$