Sparse polynomial systems with many positive solutions from bipartite simplicial complexes

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A solution $f_1(\mathbf{x}) = \cdots = f_d(\mathbf{x}) = 0$ is

- **positive** if $\mathbf{x} \in \mathbb{R}^d$ and $x_i > 0$ for all i;
- **non-degenerate** if the **jacobian matrix** of $(f_1, \ldots, f_d)$ is invertible at $\mathbf{x}$

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1. Given $\mathcal{A}$, **construct** $f_1, \ldots, f_d$ such that $f_1(\mathbf{X}) = \cdots = f_d(\mathbf{X}) = 0$ has many non-degenerate positive solutions;
2. Given $s \in \mathbb{N}$ **construct** $\mathcal{A}$ and $f_1, \ldots, f_d$ such that $|\mathcal{A}| = s$ and $f_1(\mathbf{X}) = \cdots = f_d(\mathbf{X}) = 0$ has many non-degenerate positive solutions.

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Motivation ($d = 1$): Descartes' rule of signs (1637)

The number of positive roots of a Laurent polynomial $f \in \mathbb{R}[X^{\pm 1}]$ is bounded by the number of **sign differences** between consecutive coefficients.

$\rightsquigarrow$ **all nonzero complex roots** of (squarefree) f are positive

$\Rightarrow \mathcal{A} = \{a, a+1, \ldots, a+s-1\}$ for some $a \in \mathbb{Z}$.

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Viro's method (70s): effective construction of **real hypersurfaces** with prescribed topology and support.

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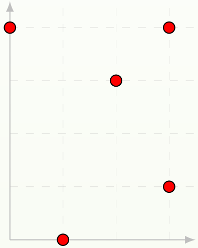

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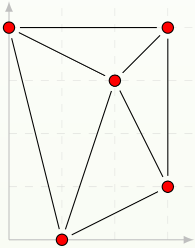

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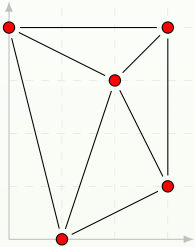

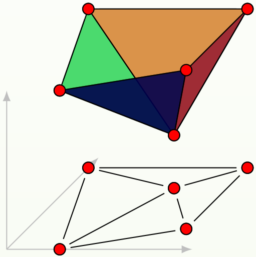

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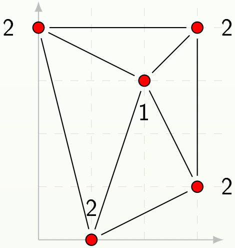

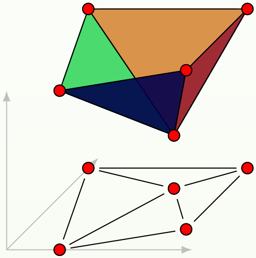

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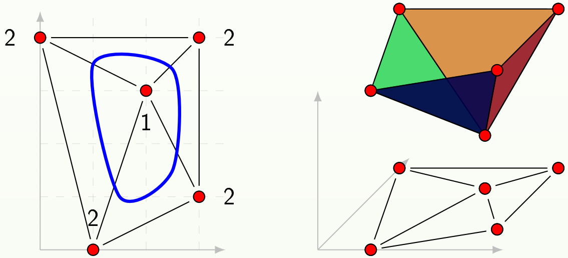

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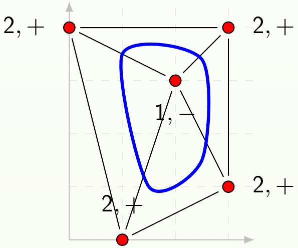

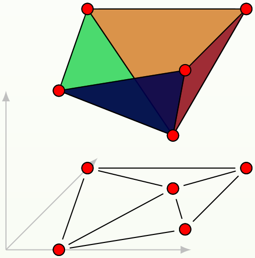

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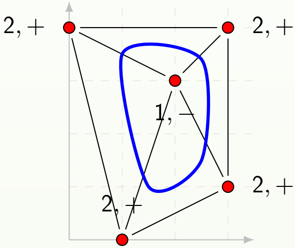

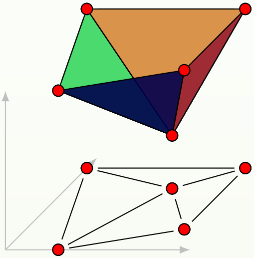

For $t > 0$ sufficiently small, the curve
$t^2 X - t X^2 Y^3 + t^2 Y^3 + t^2 X^3 Y + t^2 X^3 Y^4$ is homeomorphic to a circle in $\mathbb{R}^2_{>0}$.

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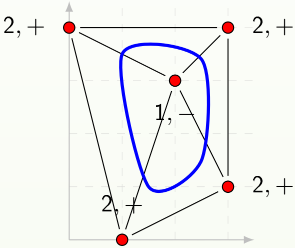

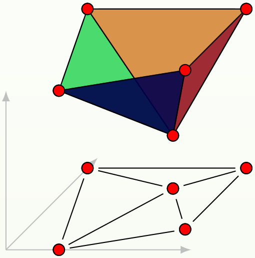

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$t^2X - tX^2Y^3 + t^2Y^3 + t^2X^3Y + t^2X^3Y^4$ is homeomorphic to a circle in $\mathbb{R}^2_{>0}$.

Considered as one of the roots of **tropical geometry**.
Extensions for **complete intersections** by Bihan, Sturmfels, Itenberg/Roy.

Main results

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Theorem

If a point configuration in $\mathbb{Z}^d$ admits a **regular, balanced, and unimodular triangulation**, then there exists a **maximally positive system** with the associated support (+ construction).

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Fewnomial systems

There exists a system of 5 equations in 5 variables, involving 11 monomials, with at least 38 positive solutions (+ construction).

$\rightsquigarrow$ under some assumption, then there exist systems of d equations in d variables, involving at most $2d+1$ monomials and having asymptotically

$$\frac{(\sqrt{2}+1)^d}{\sqrt{d}} \cdot \frac{2^{1/4}(4+2\sqrt{2})}{(12-8\sqrt{2})\sqrt{\pi}}$$

positive non-degenerate solutions.

Related works

Sturmfels '94

If a point configuration $\mathcal{A}$ in $\mathbb{Z}^d$ admits a **regular unimodular triangulation**, then there exist systems with support $\mathcal{A}$ such that all toric complex solutions are real.

This work: if **balanced**, then the solutions can be made positive.

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Real solutions of **fewnomial systems**: Bihan, Grenet, Koiran, Phillipson, Portier, Rojas, Roy, Sottile, Sturmfels, Tavenas,...

Balanced simplicial complexes: Izmestiev, Joswig, Stanley, Witte, Ziegler,...

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Bihan's conjecture

If $\mathcal{A} \subset \mathbb{Z}^d$ is the support of a **maximally positive** polynomial system, then it has a basis of **affine relations** whose coefficients are in $\{-2, -1, 0, 1, 2\}$.

Affine relation: $(b_\alpha)_{\alpha \in \mathcal{A}} \in \mathbb{Z}^{|\mathcal{A}|}$ s.t. $\sum_{\alpha \in \mathcal{A}} b_\alpha \alpha = 0$ and $\sum_{\alpha \in \mathcal{A}} b_\alpha = 0$.

Linear systems

C: full rank $d \times (d+1)$ **real matrix**.
What are the conditions on A such that

$$C \cdot \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_d \\ 1 \end{bmatrix} = 0$$

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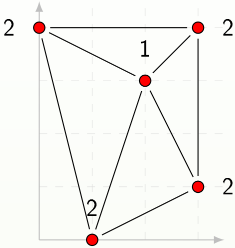
Up to an invertible monomial map, extends to any vector of $d+1$ monomials s.t. the convex hull of the exponent vectors is a d-simplex.

Variant of Viro's method



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has 4 **positive sols** for $t > 0$ sufficiently small.

Positively decorable simplicial complex

A simplicial complex $\Gamma \subset \mathbb{R}^d$ on s vertices is called **positively decorable**, if there exists a $d \times s$ matrix C (with columns indexed by vertices of Γ) such every $d \times (d+1)$ submatrix corresponding to a d-simplex has full rank and has **a positive kernel vector**.

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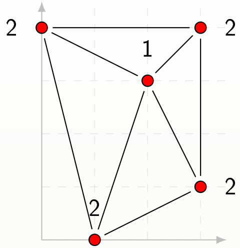

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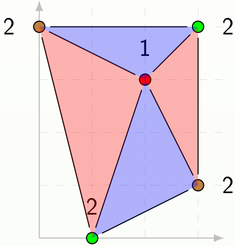
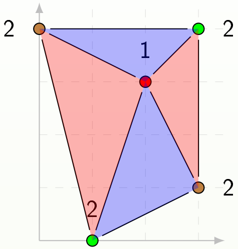
Theorem

Let $\mathcal{A} \subset \mathbb{Z}^n$ be a finite point configuration, $\Gamma \subset \mathbb{R}^d$ a simplicial complex included in a regular triangulation of Γ. If Γ is **positively decorated**, then for $t > 0$ sufficiently small, the number of positive sols of the associated system is bounded below by the number of d-simplices.



Balanced complex: 1-skeleton is $(d+1)$-coloriable
Bipartite complex: d-simplices are 2-coloriable

Balanced and bipartite simplicial complexes



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balanced	$\Rightarrow$	**positively decorable**	$\Rightarrow$	**bipartite**
easy to check		not easy to check		easy to check

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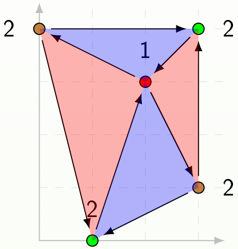

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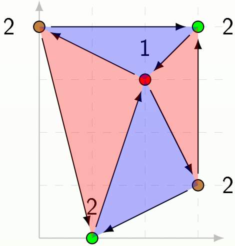

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Question

bipartite $\overset{?}{\Rightarrow}$ **positively decorable**?

Regular balanced unimodular triangulations

Theorem

If a finite point configuration $\mathcal{A} \subset \mathbb{R}^d$ admits a **regular balanced unimodular triangulation**, then there exists a **maximally positive system** with support $\mathcal{A}$.

Proof.

Kouchnirenko's theorem. □

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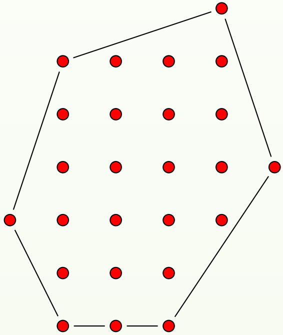

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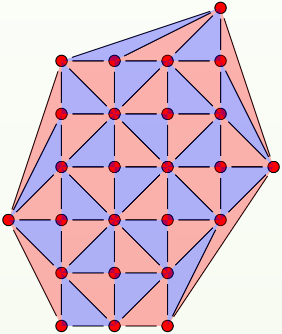

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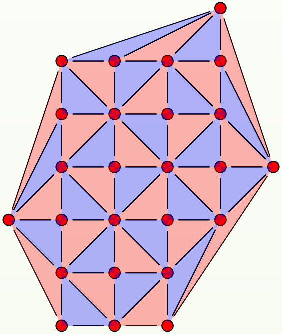

Holds true for several classical families of $\mathcal{A}$:

- order polytopes (*e.g.* multilinear systems)
- multi-homogeneous systems
- the hypersimplex

All of them satisfy **Bihan's conjecture**.

Fewnomial systems

Khovanskii's problem

Given $d, k \in \mathbb{N}$, how many non-degenerate positive sols for real systems of d equations, d unknowns involving at most $d + k + 1$ monomials?
$\Xi_{d,k}$: max of nb. of **positive non-degenerate sols** over all such systems.

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Bihan/Sottile: $\Xi_{d,k} \leq \frac{e^2+3}{4} 2^{\binom{k}{2}} d^k$.

Bihan/Rojas/Sottile: $\Xi_{d,k} \geq (\lfloor d/k \rfloor + 1)^k$

Univariate polynomials with disjoint variables: $\Xi_{d,k} \geq (\lfloor k/d \rfloor + 1)^d$.

Problem: Does there exist a system with $d = k$ and more than 2^d positive solutions?

Simplicial complex supported on the cyclic polytope

Cyclic polytope $C(s, d)$: convex hull of s points $(a_i, a_i^2, \ldots, a_i^d) \in \mathbb{R}^d$.

A bipartite simplicial complex with many simplices

We propose to use a **bipartite simplicial complex** included in a triangulation of the cyclic polytope $C(2d + 1, d)$.
As d grows, this simplicial complex has

$$\Theta\left(\frac{(\sqrt{2}+1)^d}{\sqrt{d}}\right)$$

simplices of dimension d.

but not balanced! Needs **computational methods** to positively decorate it.

Computational aspects: positive matrix completion

How to **positively decorate** a (non-balanced) simplicial complex $\Gamma \subset \mathbb{R}^d$?

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Positive matrix completion

A simplicial complex $\Gamma \subset \mathbb{R}^d$ is **positively decorable** iff there exists a $k \times \ell$ matrix M of rank $k - d$ such that

$$M_{i,j} \begin{cases} > 0 \text{ if } \mathbf{s}_i \in \Delta_j \\ = 0 \text{ otherwise,} \end{cases}$$

where $\mathbf{s}_1, \ldots, \mathbf{s}_k$ are the vertices of Γ, and $\Delta_1, \ldots, \Delta_\ell$ are its d-simplices.
If such a matrix exists, then a **basis of its left kernel** is a $d \times k$ matrix which positively decorates Γ.

Computational aspects: positive matrix completion

How to **positively decorate** a (non-balanced) simplicial complex $\Gamma \subset \mathbb{R}^d$?

Positive matrix completion

A simplicial complex $\Gamma \subset \mathbb{R}^d$ is **positively decorable** iff there exists a $k \times \ell$ matrix M of rank $k - d$ such that

$$M_{i,j} \begin{cases} > 0 \text{ if } \mathbf{s}_i \in \Delta_j \\ = 0 \text{ otherwise,} \end{cases}$$

where $\mathbf{s}_1, \ldots, \mathbf{s}_k$ are the vertices of Γ, and $\Delta_1, \ldots, \Delta_\ell$ are its d-simplices.
If such a matrix exists, then a **basis of its left kernel** is a $d \times k$ matrix which positively decorates Γ.

Cyclic polytope + NewtonSLRA (Schost/S.)
$\rightsquigarrow$ a system of 5 eqs, 5 unknowns, 11 monomials, and 38 positive solutions!

$$\Rightarrow \Xi_{5,5} \geq 38.$$

$$\text{(previously, } 32 \leq \Xi_{5,5} \leq 8311244\text{).}$$

A system with 38 positive solutions

$$C = \begin{bmatrix} \frac{14036}{26031} & \frac{-29047}{45845} & \frac{22485}{134218} & \frac{-20647}{80496} & \frac{14312}{69515} & \frac{-39015}{127243} & \frac{-6739}{42098} & \frac{19359}{360623} & \frac{16000}{83529} & \frac{1804}{131469} & \frac{4862}{44061} \\ \frac{19937}{61149} & \frac{-8379}{77942} & \frac{-2105}{18949} & \frac{5635}{122379} & \frac{9229}{59989} & \frac{5391}{113671} & \frac{17593}{33547} & \frac{-50525}{112808} & \frac{-13843}{33458} & \frac{18357}{116882} & \frac{-54686}{132521} \\ \frac{6391}{94296} & \frac{-3329}{144100} & \frac{7957}{156078} & \frac{-5685}{48451} & \frac{-14459}{74653} & \frac{30218}{245615} & \frac{-12227}{25927} & \frac{49127}{145204} & \frac{-14117}{47609} & \frac{29515}{59658} & \frac{-42328}{83609} \\ \frac{-12249}{145219} & \frac{-13663}{97873} & \frac{-25831}{90582} & \frac{26287}{33739} & \frac{6818}{23407} & \frac{-14579}{44765} & \frac{-11126}{58889} & \frac{2247}{122770} & \frac{11139}{100537} & \frac{14421}{74818} & \frac{-60016}{644607} \\ \frac{15984}{47945} & \frac{-22523}{72834} & \frac{-10734}{41165} & \frac{8531}{24837} & \frac{-21257}{47591} & \frac{22017}{37075} & \frac{5346}{284353} & \frac{19757}{194173} & \frac{5740}{83029} & \frac{-62271}{466111} & \frac{5591}{37902} \end{bmatrix}$$

$$C \cdot \begin{bmatrix} 1 \\ t\, X_1 X_2 X_3 X_4 X_5 \\ t^{2^6} X_1^{2} X_2^{2^2} X_3^{2^3} X_4^{2^4} X_5^{2^5} \\ t^{3^6} X_1^{3} X_2^{3^2} X_3^{3^3} X_4^{3^4} X_5^{3^5} \\ \vdots \\ t^{10^6} X_1^{10} X_2^{10^2} X_3^{10^3} X_4^{10^4} X_5^{10^5} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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There exist $\mathcal{A}$ s.t. the max nb. of pos. sols cannot be reached by this method.
Restricted at the moment to unmixed systems.

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$\rightsquigarrow$ if yes, then $\limsup(\Xi_{d,d})^{1/d} \geq \sqrt{2}+1$
In general, existence of a bipartite simpl. complex which is not decorable?
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"for $t > 0$ sufficiently small": explicit t_0?
Bihan's conjecture.

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Computations:

Given a finite set of points $\mathcal{A}$ in $\mathbb{Z}^d$, compute (if it exists) a regular unimodular triangulation of its convex hull.
If unimodular is not possible, find a bipartite simplicial complex with vertices $\mathcal{A}$ with as many d-simplices a possible.
(Hybrid symbolic-numeric) computational tools for the positive matrix completion problem.

Thank you!

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