Algebraic classification related to contrast optimization for MRI

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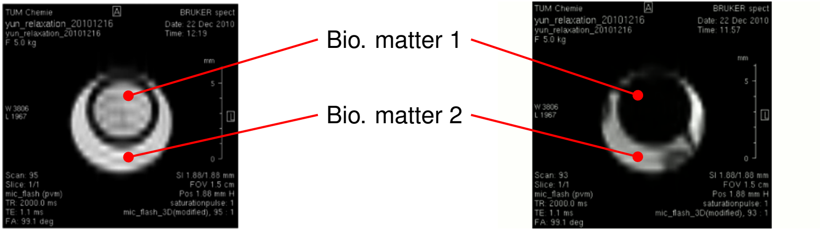
Physical problem

(N)MRI = (Nuclear) Magnetic Resonance Imagery

1. apply a magnetic field to a body
2. measure the radio waves emitted in reaction

Optimize the contrast = be able to distinguish two biological matters from this measure



Bad contrast Good contrast

Known methods:

- inject contrast agents to the patient: potentially toxic
- make the field variable to exploit differences in relaxation times
  $\Longrightarrow$ requires finding optimal settings

(Images: Pr. Steffen Glaser, Tech. Univ. München)

Numerical approach

The Bloch equations

$$\begin{cases} \dot{y}_i &= -\Gamma_i y_i - u_x z_i \\ \dot{z}_i &= -\gamma_i (1 - z_i) + u_x y_i \end{cases}$$

$(i = 1, 2)$

Saturation method

Find a path u_x so that after some time T:

- matter 1 saturated: $y_1(T) = z_1(T) = 0$
- matter 2 “maximized”: $|(y_2(T), z_2(T))|$ maximal

Glaser’s team, 2012 : Control Theory method

- Numerical method to find a path towards a saturated system = solution u_x
- Already used in some specific cases for the MRI, here applied in full generality

Numerical approach... and computational problem

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Problem:

- The complexity of the path u_x is not bounded

Goal:

- Classify singular trajectories for the control
- Obtain control policies for the contrast problem

This classification problem can be modelled with polynomials!

System

- $M := \begin{pmatrix} -\Gamma_1 y_1 & -z_1 - 1 & -\Gamma_1 + (\gamma_1 - \Gamma_1)\, z_1 & (2\,\gamma_1 - 2\,\Gamma_1)\, y_1 \\ -\gamma_1 z_1 & y_1 & (\gamma_1 - \Gamma_1)\, y_1 & 2\,\Gamma_1 - \gamma_1 - (2\,\gamma_1 - 2\,\Gamma_1)\, z_1 \\ -\Gamma_2 y_2 & -z_2 - 1 & -\Gamma_2 + (\gamma_2 - \Gamma_2)\, z_2 & (2\,\gamma_2 - 2\,\Gamma_2)\, y_2 \\ -\gamma_2 z_2 & y_2 & (\gamma_2 - \Gamma_2)\, y_2 & 2\,\Gamma_2 - \gamma_2 - (2\,\gamma_2 - 2\,\Gamma_2)\, z_2 \end{pmatrix}$
- $D := \det(M)$

Problem

Find all zeroes of D which are singular in (y_1, y_2, z_1, z_2)

Equivalent formulation

Find the zeroes of

$$\left\langle D, \frac{\partial D}{\partial y_1}, \frac{\partial D}{\partial y_2}, \frac{\partial D}{\partial z_1}, \frac{\partial D}{\partial z_2} \right\rangle$$

- Method described in [Bonnard et al. 2013]
- Proof of concept: they used this method to solve the problem for the 4 experimental settings serving as examples to the saturation method
- Question : solutions in full generality?

Method

Obvious method?

Compute a Gröbner basis of this system

- Works in theory: method used in [Bonnard et al. 2013]
- Impracticable in full generality

This work

Decomposition into simpler problems

- easy simplifications (e.g. $\gamma_1 = 1$)
- specific physical cases: e.g. matter 1 is water $\iff \Gamma_1 = \gamma_1$
- specific structure of the system
- systematic study of factorizations

What is "simpler"?

- More constraints: study $I + \langle f \rangle \iff$ study $V(I) \cap V(f)$
- Less components: study $I + \langle Uf - 1 \rangle \iff$ study $V(I) \setminus V(f)$

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Typical example

If I contains $f \cdot g$, we can decompose the system into:

- either $f = 0 \to$ add f to the system
- or $f \neq 0$ and $g = 0 \to$ add $U f - 1$ and g to the system

First decomposition: rank of the matrix

We split the problem depending on the rank of the matrix:

Base problem

$\mathrm{rank}(M) < 3$ $\quad$ $\mathrm{rank}(M) = 3$

Why the rank? Because of...

Theorem

Consider

$$M = (P_{i,j}(\mathbf{X}))_{1 \le i,j \le n}$$

Then generically:

$$\det(M) \text{ singular} \implies \mathrm{rank}(M) < n - 1$$

For our specific matrix, we do not know if the theorem applies.

$\implies$ We need to consider both branches.

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The case $\text{rank}(M) < 3$: classification

$$\left\langle D, \frac{\partial D}{\partial y_1}, \frac{\partial D}{\partial y_2}, \frac{\partial D}{\partial z_1}, \frac{\partial D}{\partial z_2}, \text{3-minors of } M \right\rangle \text{ contains } P = \sum_{d=0}^{4} a_d(\Gamma_1, \Gamma_2, \gamma_2) y_2^{2d}$$

We classify depending on the number of roots of P in $Y_2 = y_2^2$:

- First bound: degree of P
- Need to handle multiple roots (for example using discriminants)

$\vdots$

$\text{rank}(M) < 3, \; Y_2 = y_2^2$

No solution in Y_2

$a_1 = \cdots = a_4 = 0$
$a_0 \neq 0$

1 solution in Y_2

$a_2 = \cdots = a_4 = 0$
$a_1 \neq 0$

$a_3 = a_4 = 0$
$a_2 \neq 0$
$\text{disc}(P) = 0$

$\vdots \quad \vdots$

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The case $\mathrm{rank}(M) = n - 1$: incidence varieties

Theorem

Consider $M = (P_{i,j}(\mathbf{X}))_{1 \le i,j \le n}$ and let $\mathcal{I}$ be the incidence variety defined by

$$\begin{bmatrix} & & \\ & M & \\ & & \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

If $(\mathbf{x})$ is a point of $V(\det(M))$, then:

- there exists a non-zero vector $\Lambda = (\lambda_i)$ such that $(\mathbf{x}, \Lambda) \in \mathcal{I}$

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If $(\mathbf{x})$ is a singular point of $V(\det(M))$ such that $M(\mathbf{x})$ has rank $n-1$, then:

- there exists a non-zero vector $\Lambda = (\lambda_i)$ such that $(\mathbf{x}, \Lambda) \in \mathcal{I}$, and
- Λ is unique up to scalar multiplication, and
- $(\mathbf{x}, \Lambda)$ is a singular point of $\mathcal{I}$ w.r.t. $\mathbf{X}$

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$$\left\langle D, \frac{\partial D}{\partial y_i, z_i}, \underbrace{\boxed{M \cdot \Lambda, \frac{\partial M \cdot \Lambda}{\partial y_i, z_i}}}_{\text{Incidence variety}}, \underbrace{\boxed{\mathcal{M}_k \neq 0}}_{\substack{\text{Non-zero } 3 \times 3 \text{ minor} \\ \text{(16 choices)}}}, \underbrace{\boxed{\lambda_i = 1}}_{\substack{\text{Non-zero coordinate} \\ \text{(4 choices)}}} \right\rangle_{\substack{1 \le k \le 16 \\ 1 \le i \le 4}}$$

Overview of the classification

Base problem

rank(M) < 3

$Y_2 \leftarrow y_2^2$

6 branches according to
the number of roots of P

Branches according to
the degree of P
and the roots multiplicity

More branches from factorizations

rank(M) $= 3$

16 branches according to
the nonzero minor

4 branches according to
the non-zero coordinate in Λ

More branches from factorizations

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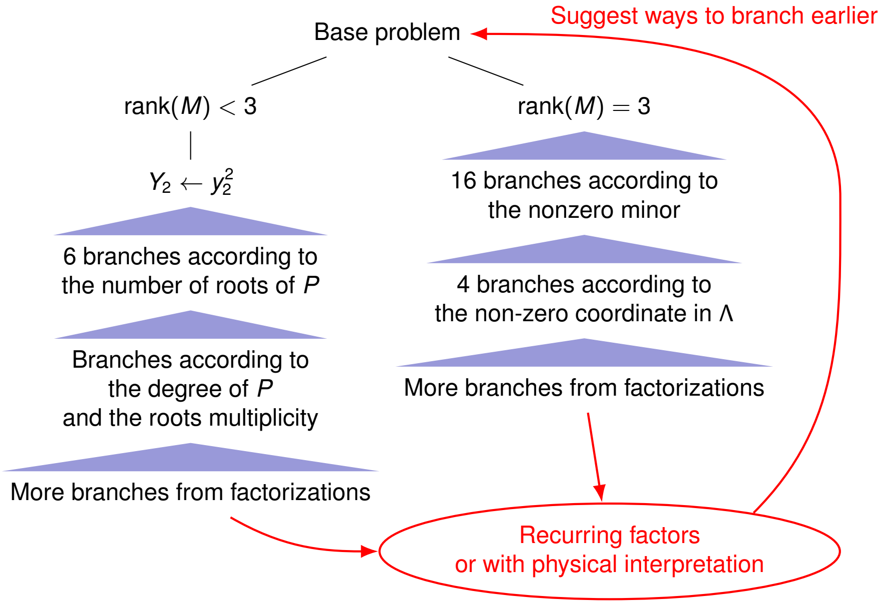

Conclusion, perspectives

What has been done?

Classification of singular trajectories for the saturation control

- Exhaustive classification in some particular cases
- Some branches entirely explored in full generality

Still work in progress

- Some branches not solved yet in full generality
- More physical constraints have to be taken into account
- Specific physical cases do not necessarily appear in the classification

Applications

- Identification of the situations where the saturation method may fail
- New control policies trying to avoid these points

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Thank you for your attention!

Merci pour votre attention!