Minimal unique palindromic substrings after single-character substitution

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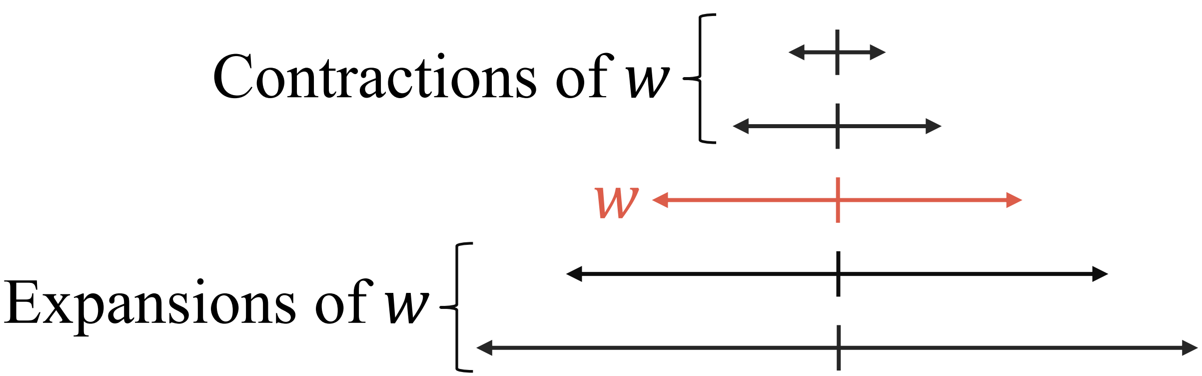
Palindromes

Palindrome is a string that reads the same forward and backward.

For a palindromic substring w of a string T,
a palindrome whose center is the same as that of w is called

- a **contraction** of w if it is shorter than w, and
- an **expansion** of w if it is longer than w.

T = a b b a a b b b a a b a b a b a b



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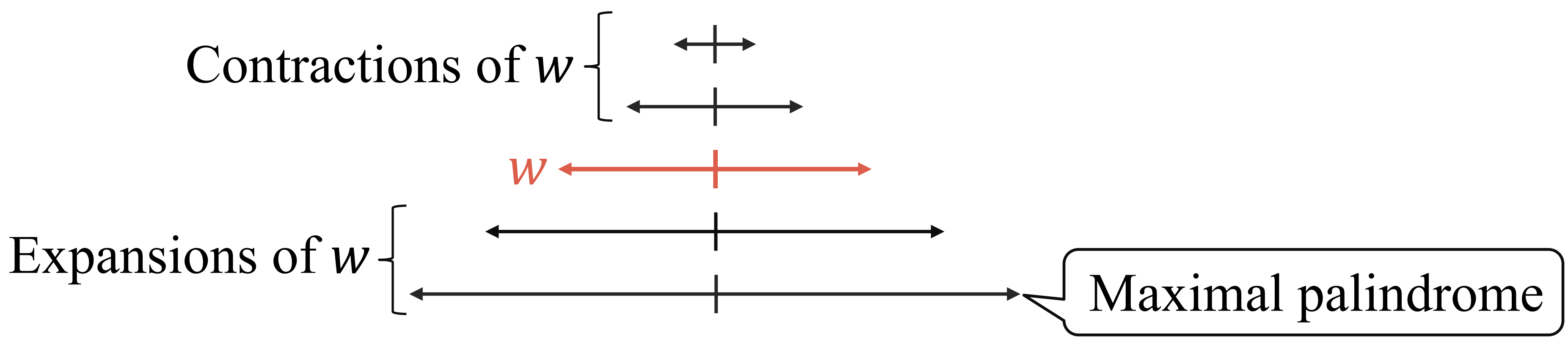
If there are no expansions of u, u is called a maximal palindrome.

T = a b b a a b b b a a b a b a b a b



Minimal Unique Palindromic Substring

Definition [Inoue+, '18]

A palindromic substring $T[i..j]$ of a string T is a **minimal unique palindromic substring (MUPS)** of T if $T[i..j]$ is unique in T and $T[i+1..j-1]$ is repeating in T.

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T = a b **b a a b** b **b a a b** a b a b a b

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T = a b b a a b b b a a b a b a b a b

Theorem [Inoue+, '18]
For any string T of length n, $|\mathrm{MUPS}(T)| \leq n$.
$\mathrm{MUPS}(T)$ can be computed in $O(n)$ time.

$\mathrm{MUPS}(T)$: the set of MUPSs of a string T.

Our problem

> **Problem** (*MUPS_AFTER_EDIT*)
> Preprocessing input: string T
> Query input: single character substitution $\langle i, s \rangle$
> Query output: $\mathrm{MUPS}(T) \triangle \mathrm{MUPS}(T')$ (T': the edited string)

Substituting $T[i]$ with a character s.

Query input: $\langle i, s \rangle = \langle 7, \mathrm{a} \rangle$

i

T = a b b a a b $\boxed{\texttt{b}}$ b a a b a b a b

T' = a b b a a b $\boxed{\texttt{a}}$ b a a b a b a b

Query output: abba, bbb, bbaabb, bb, aababaa, abaaba

Removed Added

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Substituting $T[i]$ with a character s.

Query input: $\langle i, s \rangle = \langle 9, \mathrm{b} \rangle$

i

T = a b b a a b b b $\boxed{\texttt{a}}$ a b a b a b

T' = a b b a a b b b $\boxed{\texttt{b}}$ a b a b a b

Query output: bbb, bbaabb, babab, aa, bbbb

Removed Added

Related work for “after one-edit” problems

	Static
Longest common substring	$O(n)$ time [Weiner, ’73]
Longest palindrome	$O(n)$ time [Manacher, ’75]
Longest Lyndon substring	$O(n)$ time [Duval, ’83]

Related work for "after one-edit" problems

	Static	One edit
Longest common substring	$O(n)$ time [Weiner, '73]	$\tilde{O}(n)$ space $\tilde{O}(1)$ time [Amir+, '17] [Abedin+, '18]
Longest palindrome	$O(n)$ time [Manacher, '75]	$O(n)$ space $\tilde{O}(1)$ time [Funakoshi+, '21]
Longest Lyndon substring	$O(n)$ time [Duval, '83]	$O(n)$ space $\tilde{O}(1)$ time [Urabe+, '18]

$\tilde{O}(f(n)) = O(f(n)\ \mathrm{polylog}(n))$

Related work for “after one-edit” problems

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Longest palindrome	$O(n)$ time [Manacher, ’75]	$O(n)$ space $\tilde{O}(1)$ time [Funakoshi+, ’21]	$O(n)$ space $\tilde{O}(1)$ time [Amir & Boneh, ’19]
Longest Lyndon substring	$O(n)$ time [Duval, ’83]	$O(n)$ space $\tilde{O}(1)$ time [Urabe+, ’18]	$O(n)$ space $\tilde{O}(n^{1/2})$ time [Amir+, ’19]*

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MUPS	$O(n)$ time [Inoue+, ’18]	$O(n)$ space $\tilde{O}(1)$ time **[This work]**	-

$\tilde{O}(f(n)) = O(f(n)\ \mathrm{polylog}(n))$

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Main results

Theorem 1

The number $\boldsymbol{d}$ of changes of MUPSs is $\boldsymbol{O(\log n)}$ after single character substitution.

Theorem 2

MUPS_AFTER_EDIT can be solved in the following query time after $O(n)$ time and space preprocessing:

	Alphabet size σ	Query time
Algorithm 1 (for large σ)	$O(\mathrm{poly}(n))$	$O(\log \sigma + (\log \log n)^2 + d)$
	$O(n)$	$O((\log \log n)^2 + d)$
Algorithm 2 (for small σ)	$O(\log n)$	$O(\log \log n + d)$
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of changes of MUPSs

> **Theorem 1**
> The number $\boldsymbol{d}$ of changes of MUPSs is $\boldsymbol{O(\log n)}$.

We show # of MUPSs to be **removed** is $O(\log n)$.

We categorize MUPSs to be removed into three types:

- Type 1: covers the editing position i.
- Type 2: does not cover i and is repeating in T'.
- Type 3: does not cover i and is unique but not minimal in T'.

$$T \;=\; \mathtt{a\ b\ b\ a\ a\ b}\ \overset{i}{\boxed{\mathtt{b}}}\ \mathtt{b\ a\ a\ b\ a\ b\ a\ b}$$

Type 3 Type 2

Type 1

$$T' \;=\; \mathtt{a\ b\ b\ a\ a\ b}\ \boxed{\mathtt{a}}\ \mathtt{b\ a\ a\ b\ a\ b\ a\ b}$$

of changes of MUPSs

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We show # of MUPSs to be **removed** is $O(\log n)$.
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i

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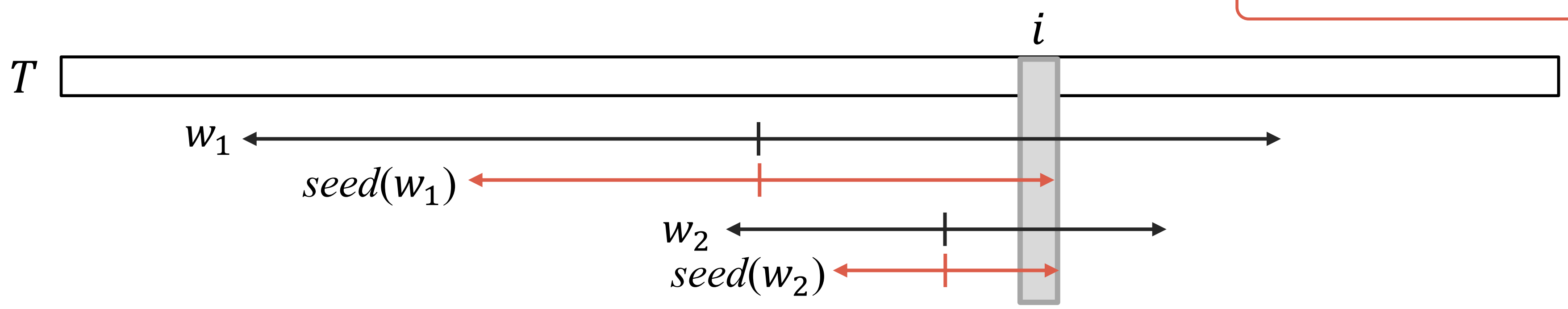
of MUPSs of Type 1

Type 1: covers i.

I focus on the MUPSs of Type 1 centered before i.

Such MUPS w is an expansion of some palindromic suffix of $T[1..i]$.

seed of MUPS w



Seeds of MUPSs are different from each other.

Thus, # of seeds is equal to # of MUPSs of Type 1.

of MUPSs of Type 1

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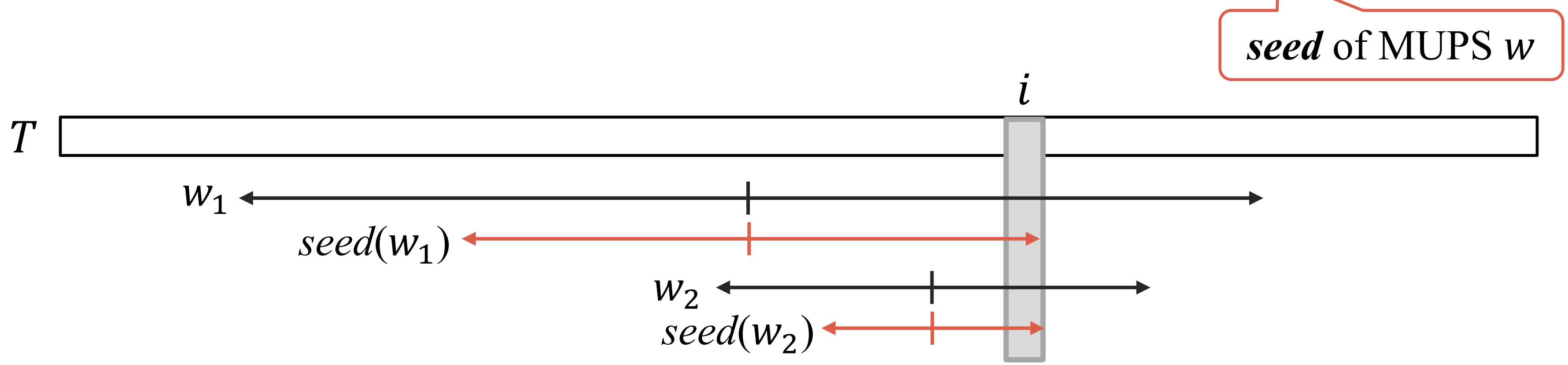

Lemma 1 [Apostolico+, ’95]

The set of palindromic suffixes of $T[1..i]$ is divided into $O(\log i)$ groups w.r.t. their smallest period.

Claim 1

The number of seeds in each group is at most two.

If Claim 1 holds, then # of MUPSs of **Type 1** is $\boldsymbol{O(\log n)}$.

Proof of Claim 1

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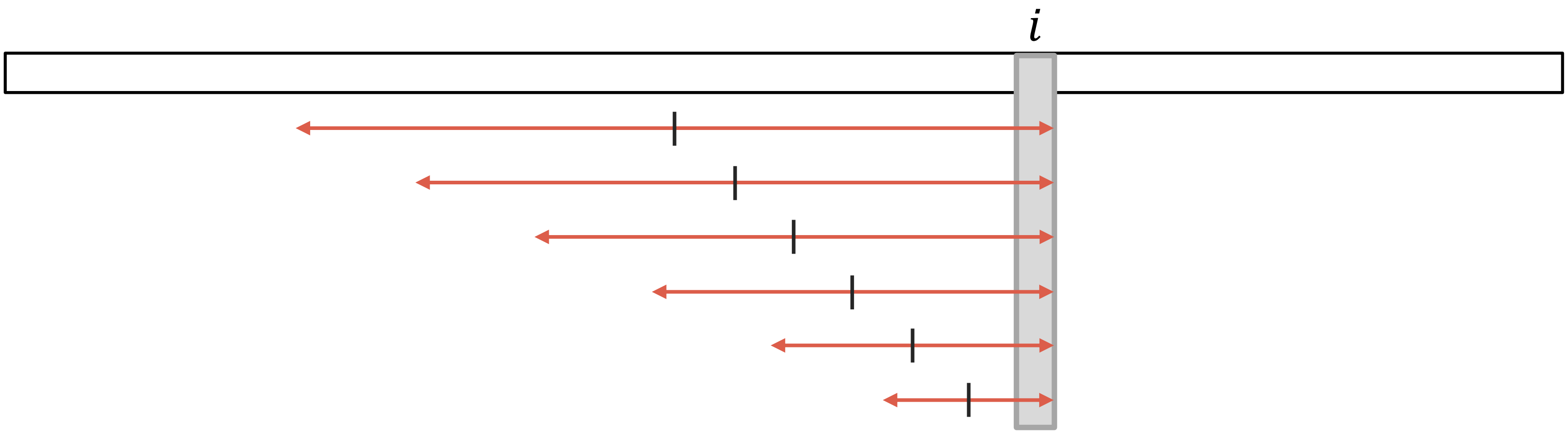

Now we consider the maximal expansions of palindromic suffixes, namely, maximal palindromes.

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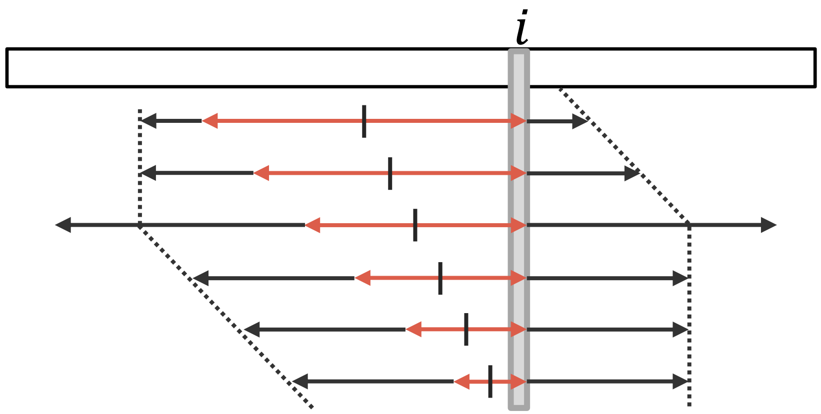
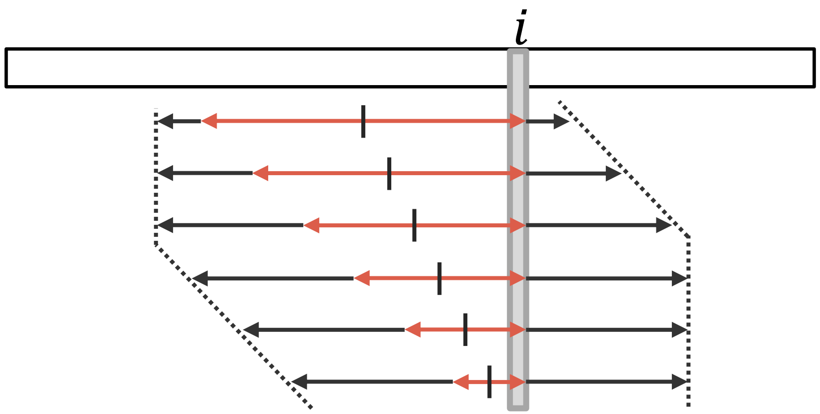


or



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Due to periodicity, the expansions shape like this figure.

Proof of Claim 1

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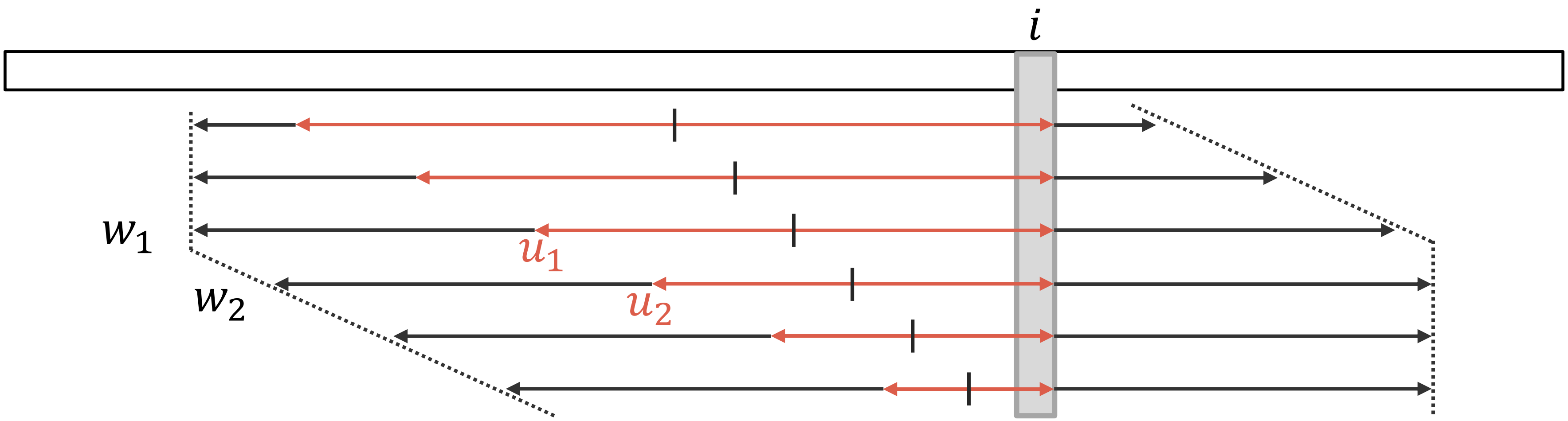


- Any palindrome which is an expansion of palindromic suffixes is contained in w_1 or w_2.
- Any expansion of palindromic suffixes excluding u_1 and u_2 is repeating.

Thus, only u_1 and u_2 can be seeds.

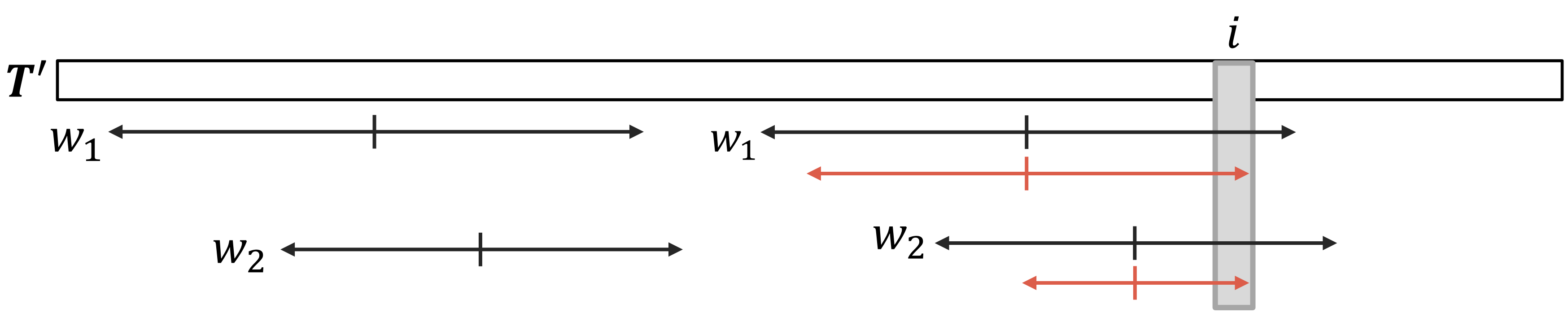
of MUPSs of Type 2

Type 2: does not cover i and is repeating in T'.

MUPS w of Type 2 has a new occurrence covering i in $\boldsymbol{T'}$.

Such an occ. is an expansion of a palindromic suffix of $T'[1..i]$.

Similar to the proof of Type-1, we reduce the problem of counting # of MUPSs of Type 2 to that of counting # of the palindromic suffixes corresponding to them.



We can prove that # of MUPSs of **Type 2** is $\boldsymbol{O(\log n)}$.

of changes of MUPSs

Similarly, we can also prove that # of MUPSs of **Type 3** is $\boldsymbol{O(\log n)}$.

From the above, # of MUPSs to be removed is $O(\log n)$.
By symmetry, # of MUPSs to be added is also $O(\log n)$.
Then we obtain Theorem 1.

Theorem 1
The number $\boldsymbol{d}$ of changes of MUPSs is $\boldsymbol{O(\log n)}$.

We have shown this upper bound is tight after submission.

Main results

Theorem 1

The number d of changes of MUPSs is $O(\log n)$ after single character substitution.

Theorem 2

MUPS_AFTER_EDIT can be solved in the following query time after $O(n)$ time and space preprocessing:

Alphabet size σ	Query time	
$O(\mathrm{poly}(n))$	$O(\log\sigma + (\log\log n)^2 + d)$	Using path-tree LCE query on EERTREE
$O(n)$	$O((\log\log n)^2 + d)$	
$O(\log n)$	$O(\log\log n + d)$	Using NCA query on Suffix Tree
$O(1)$	$O(1 + d)$	

Algorithm for $\sigma \in O(1)$

We separately compute MUPSs to be removed or added.
I will show how to compute MUPSs to be **removed**.

We use same categorizations of MUPSs to be removed:

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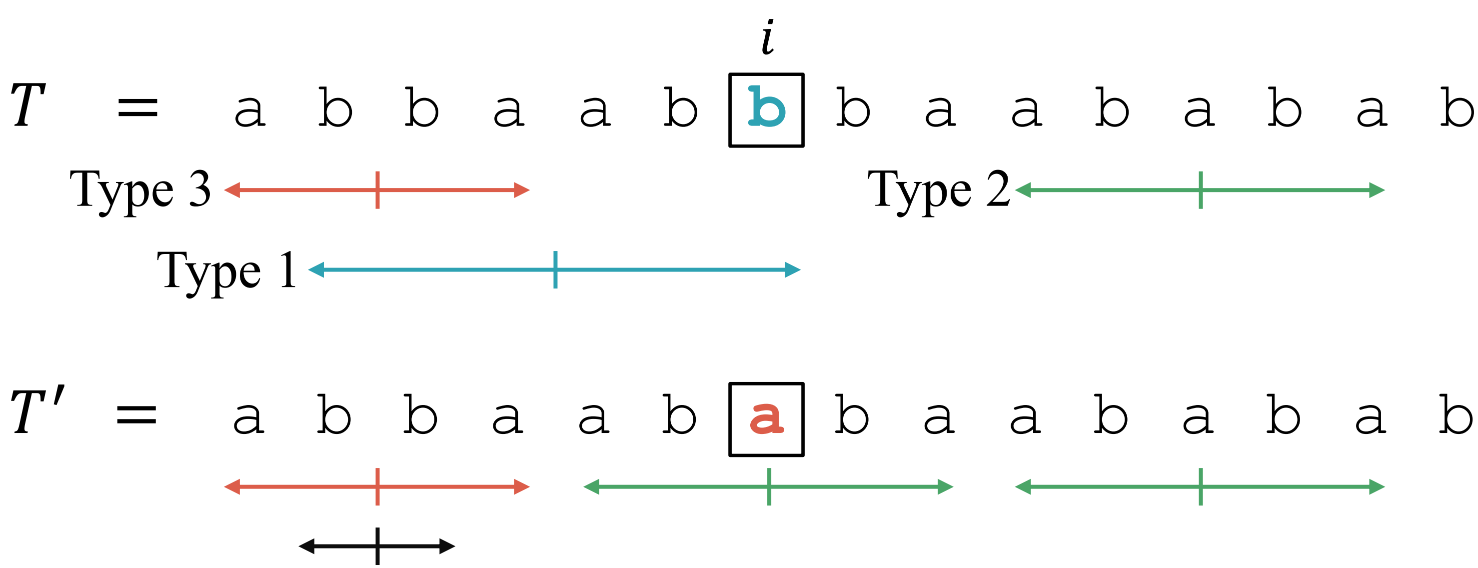

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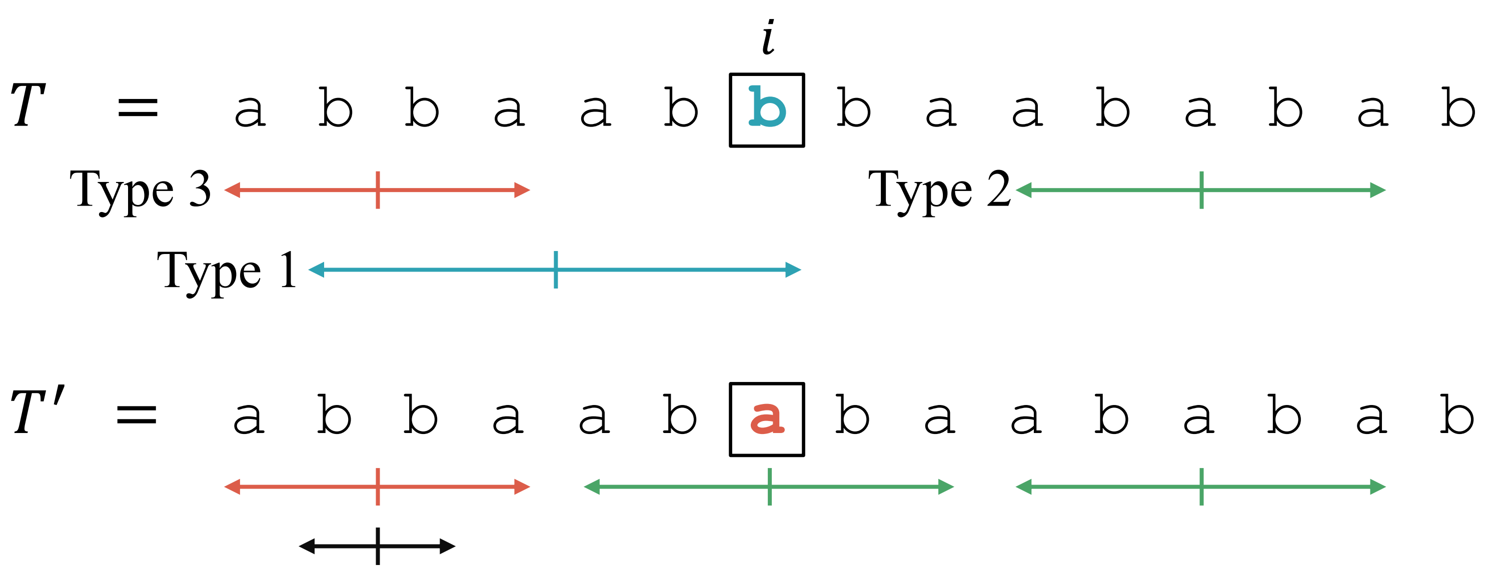


Computing MUPSs of Type 2

<u>Type 2: does not cover i and is repeating in T'.</u>

MUPS w of Type 2 satisfies the following properties in $\boldsymbol{T'}$:

- \# of occurrences of w that covers i is at least 1.
- \# of occurrences of w that does not cover i is 1.

Further, we categorize MUPSs of Type 2 into two sub-types:

2-1: at least one occurrence of w covers i by its arm.
2-2: the only occurrence of w covering i is centered at i.

i

T = a b a a b a a b c a b a a $\boxed{\texttt{c}}$ a a b c

Type 2-1

Type 2-2

T' = a b a a b a a b c a b a a $\boxed{\texttt{b}}$ a a b c

Observation for MUPSs of Type 2-1

Type 2-1: at least one occurrence of w covers i by its arm.

Let j be the starting position of an occurrence of w such that its right-arm covers i in $\boldsymbol{T'}$.

$$\begin{array}{cccccccccccccccccccc} & & & & & & & & & & j & & & & i & & & & \\ \boldsymbol{T'} & = & \mathtt{a} & \mathtt{b} & \mathtt{a} & \mathtt{a} & \mathtt{b} & \mathtt{a} & \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{a} & \mathtt{b} & \mathtt{a} & \mathtt{a} & \boxed{\mathtt{b}} & \mathtt{a} & \mathtt{a} & \mathtt{b} & \mathtt{c} \end{array}$$

w $\qquad$ w

Observation for MUPSs of Type 2-1

Type 2-1: at least one occurrence of w covers i by its arm.

Let j be the starting position of an occurrence of w such that its right-arm covers i in $\boldsymbol{T'}$.

The substring of length $|w|$ starting at j is a 1-mismatch palindrome whose left-arm matches that of w in $\boldsymbol{T}$.

$$
\begin{array}{cccccccccccccccccc}
 & & & & & & & & & & j & & & i & & & & \\
T & = & \mathtt{a} & \mathtt{b} & \mathtt{a} & \mathtt{a} & \mathtt{b} & \mathtt{a} & \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{a} & \mathtt{b} & \mathtt{a} & \mathtt{a} & \boxed{\mathtt{c}} & \mathtt{a} & \mathtt{a} & \mathtt{b} & \mathtt{c} \\
 & w & & & & & & & & & & & & & & & & \\
T' & = & \mathtt{a} & \mathtt{b} & \mathtt{a} & \mathtt{a} & \mathtt{b} & \mathtt{a} & \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{a} & \mathtt{b} & \mathtt{a} & \mathtt{a} & \boxed{\mathtt{b}} & \mathtt{a} & \mathtt{a} & \mathtt{b} & \mathtt{c} \\
 & w & & & & & & & & w & & & & & & & &
\end{array}
$$

Preprocessing for MUPSs of Type 2-1

Enumerate all 1-mismatch palindromes whose left- or right-arm matches that of some MUPS.

For each of the 1-mismatch palindromes, store the corresponding MUPS w with the key $(q, T[p])$, where q is the mismatched position and p is the position on w corresponding to q.

$$T = \texttt{a b a a}\ \overset{p}{\texttt{b}}\ \texttt{a a b c}\ \overset{j}{\texttt{a}}\ \texttt{b a a}\ \overset{q}{\boxed{\texttt{c}}}\ \texttt{a a b c}$$

w

The total time complexity is proportional to the total sum of occurrences of arms of all MUPSs.

Lemma 2
The total sum of occurrences of arms of all MUPSs is $O(n)$.

Thus, the above operations can be processed in $O(n)$ time and space.

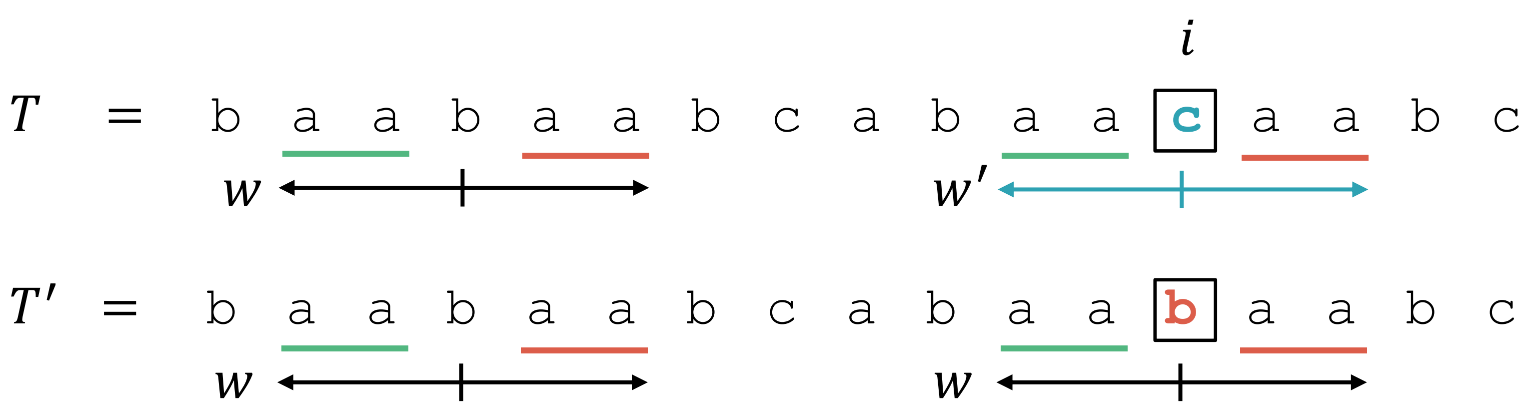
Observation for MUPS of Type 2-2

<u>Type 2-2: the only occurrence of w covering i is centered at i.</u>

There exists at most one MUPS of Type 2-2 for a query $\langle i, s \rangle$.
Also, such a MUPS w is an odd-palindrome.

If there exists such MUPS w for a query $\langle i, s \rangle$,
then there is a palindrome w' centered i such that $|w| = |w'|$ in T.
They differ only in the center character.
$\rightarrow$ The right-arm of w occurs at position $i + 1$ in both T and T'.

We design our algorithm by focusing on
occurrences of the right-arm of each MUPS in T.



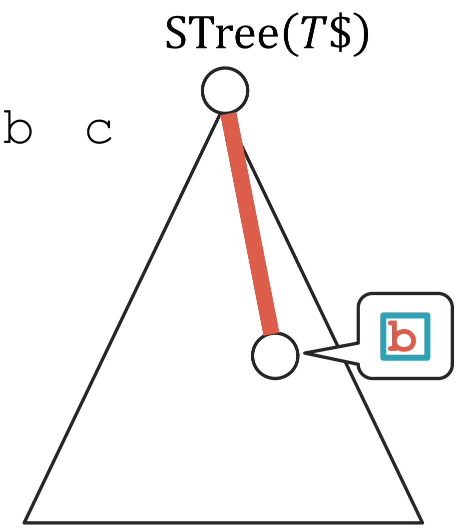
Preprocessing for MUPS of Type 2-2

We first construct the suffix tree STree(T\$).

For each odd MUPS in T, we make the locus of the right-arm explicit on STree(T\$) and label the node with the center character.

T = b a a b a a b c a b a a c a a b c

w



Preprocessing for MUPS of Type 2-2

We first construct the suffix tree STree(T\$).

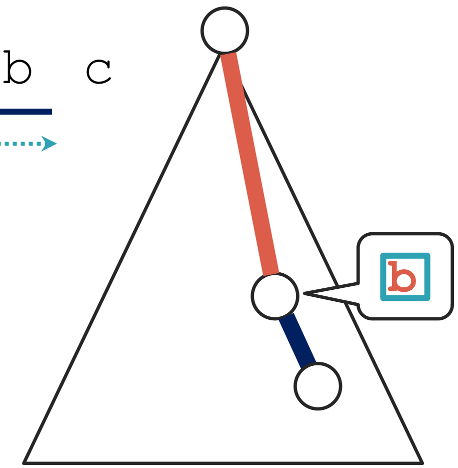
For each odd MUPS in T, we make the locus of the right-arm explicit on STree(T\$) and label the node with the center character.
Also, for each odd maximal palindrome,
we make the locus of the right-arm explicit on STree(T\$).

T = b a a $\boxed{\texttt{b}}$ a a b c a b a a c a a b c

w

odd maximal palindrome

STree(T\$)

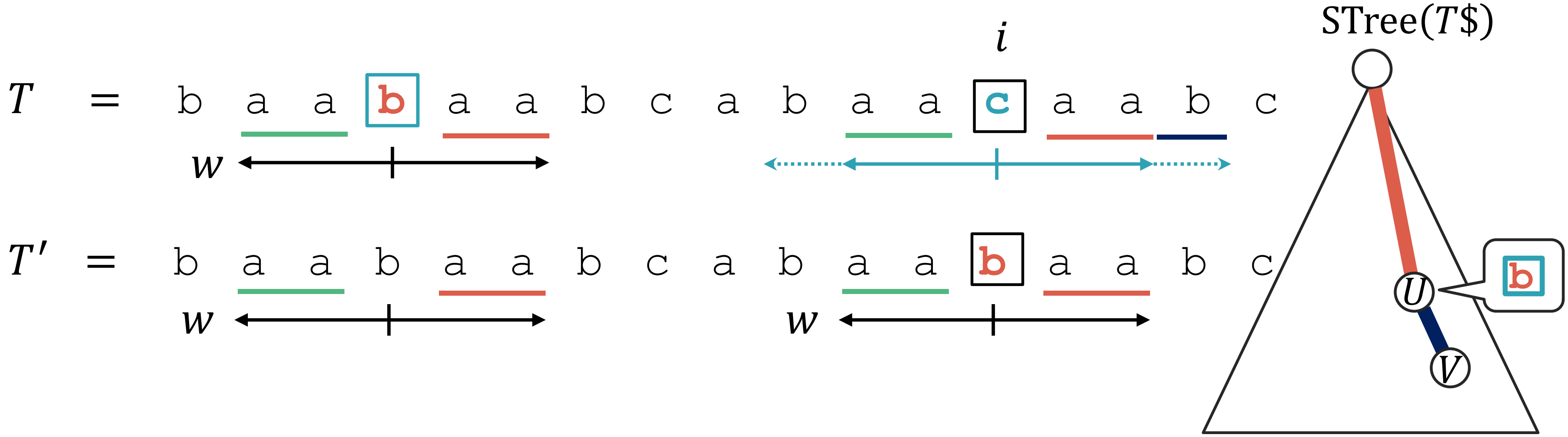


The above operations can be processed in $O(n)$ time and space by using weighted ancestor queries, Manacher's algorithms, and so on.

Query for MUPS of Type 2-2

Given a substitution $\langle i, s \rangle$,
we compute the nearest ancestor U labeled by s of the node V corresponding to the right-arm of the maximal palindrome centered at i.

If such U exists, $\mathrm{str}(U)^R \cdot s \cdot \mathrm{str}(U)$ is a MUPS of Type 2-2.



Query for MUPS of Type 2-2

Given a substitution $\langle i, s \rangle$,
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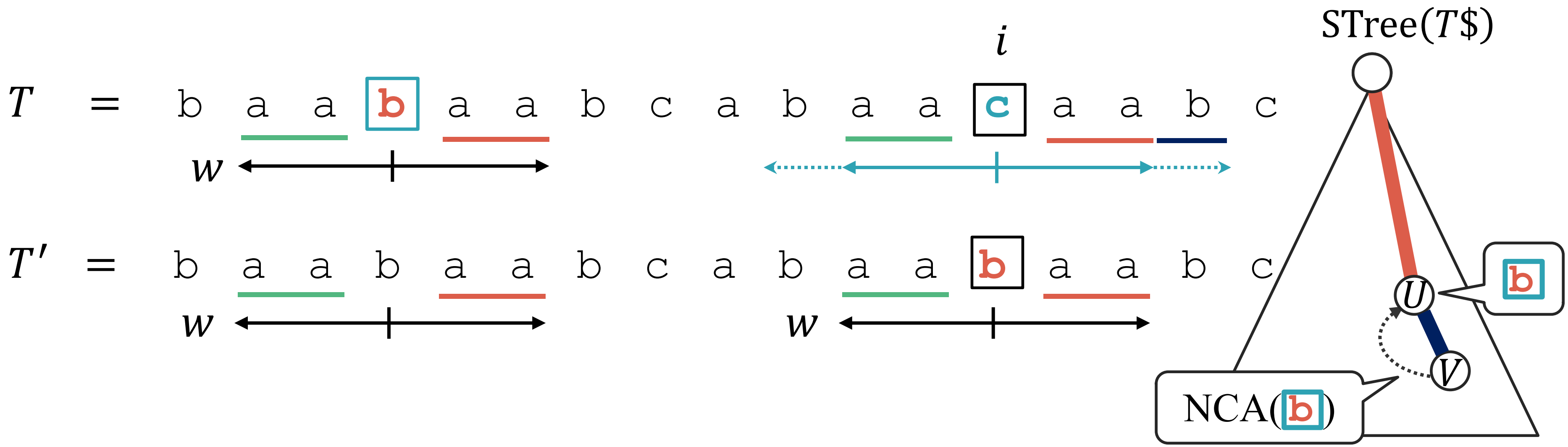
Nearest Colored Ancestor (NCA)

If such U exists, $\mathrm{str}(U)^R \cdot s \cdot \mathrm{str}(U)$ is a MUPS of Type 2-2.



If # of colors is $O(\log n)$, any NCA query can be answered in $O(1)$ time after $O(n)$-time preprocessing [Bille+, ’15][Charalampopoulos+, ’21].
Thus, the MUPS of Type 2-2 can be computed in $O(1)$ query time.

Summary and future work

Our results

- The number d of changes of MUPSs is $O(\log n)$.
- *MUPS_AFTER_EDIT* can be solved in the following query time after $O(n)$ time and space preprocessing:

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Algorithm 1 (for large σ)	$O(\mathrm{poly}(n))$	$O(\log\sigma + (\log\log n)^2 + d)$
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	$O(1)$	$O(1 + d)$

Future work

- Insertions and deletions?
- Fully dynamic algorithm?
  - Can the techniques in [Amir and Boneh, '19] be utilized?