# On stricter reachable repetitiveness measures

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- But... is it?

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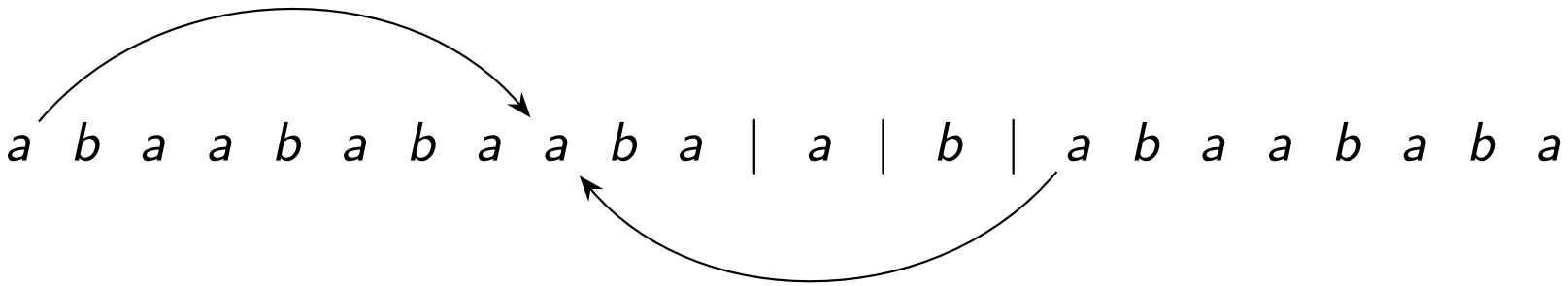
- Is it the only feature to exploit?
- We exploit **structural repetitiveness**, also known as self-similarity.
- This form of repetitiveness is not captured by the current measures.
- For Thue-Morse words, for example, we obtain a reachable representation of constant size.

# Contents

# Bidirectional macro schemes



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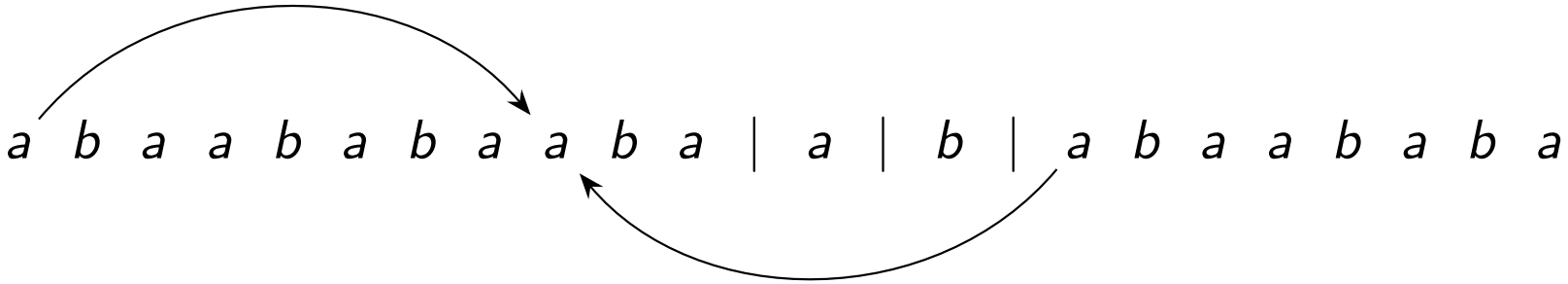


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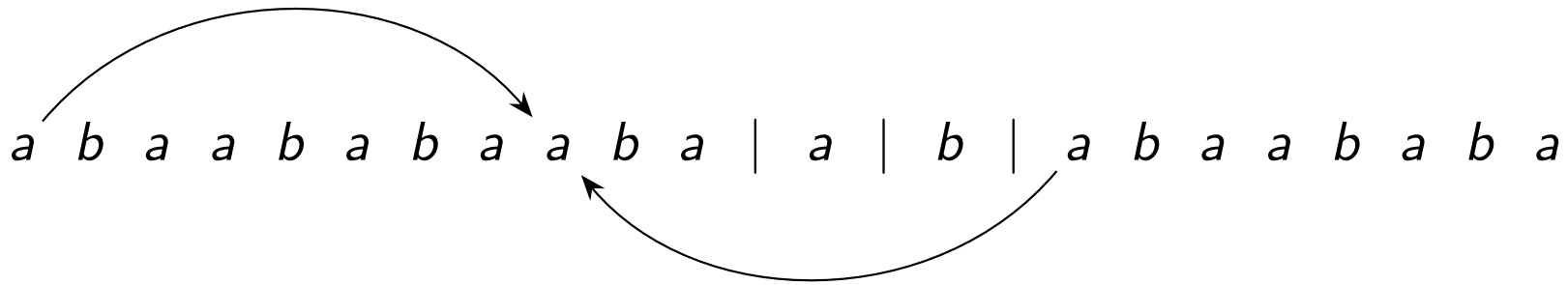


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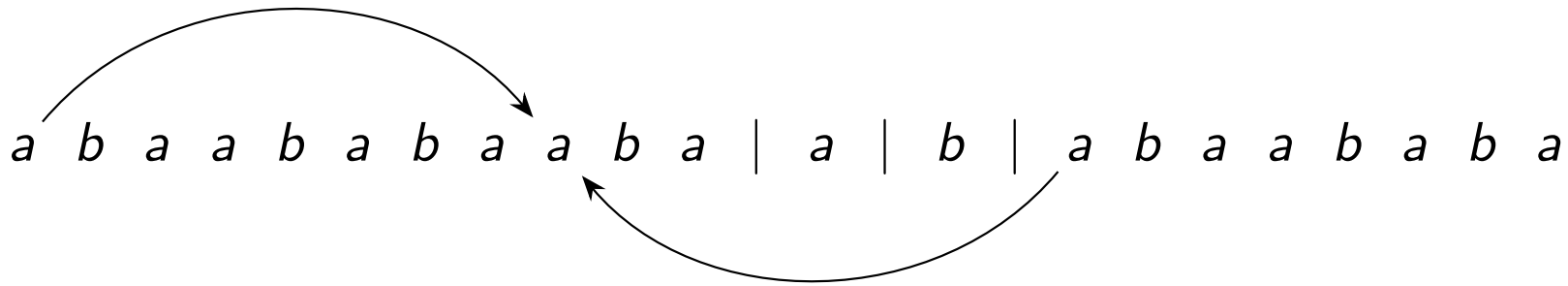


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- In the example, $B = (abaababaaba, 9), (a, \perp), (b, \perp), (abaababa, 9)$ is a minimal BMS of size 4.
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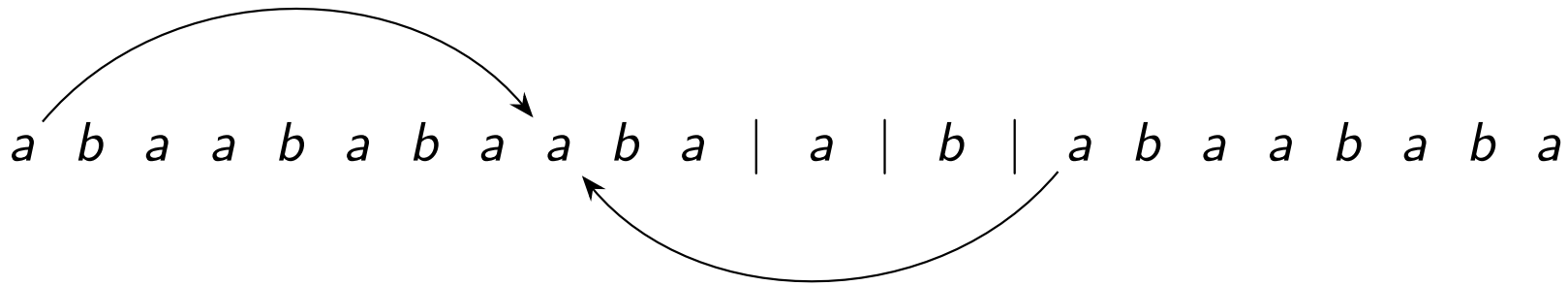


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- We store a pair of $O(\log n)$ bits per phrase, containing length and source, to recover $w$. In the example, $(11, 9), (1, a), (1, b), (8, 9)$.
- The size $b(w)$ of the smallest of them is NP-hard to compute.

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- The extraction symbols $A[i,j]$ mean to extract the explicit symbols of $exp(A)[i,j]$.
- We denote by $m$ the size of the smallest macro system generating $w$.

# Macro systems - Results

**Theorem 1**

*It always holds that* $m = O(b)$.

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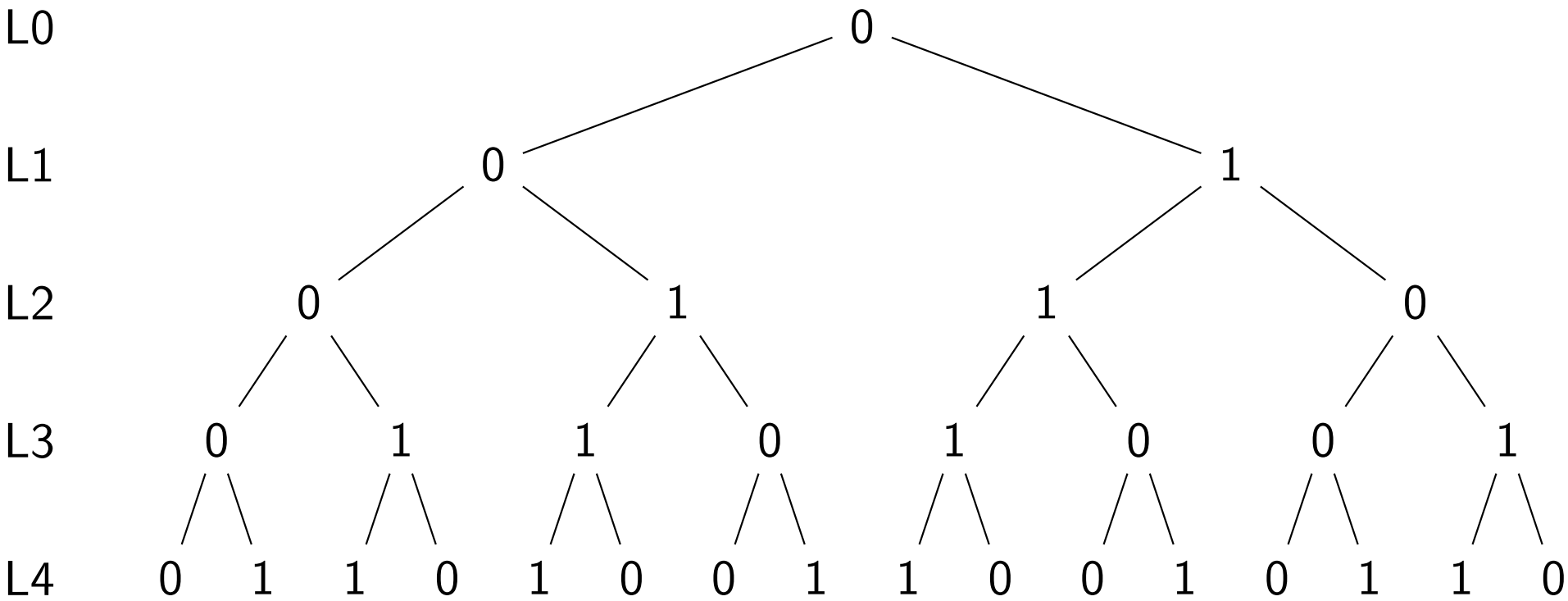
The *size* of an L-system is $|S| + \sum_{A \in V} |R(A)|$. We call $\ell$ the size of the smallest L-system generating a string $w$.

# L-systems - Example: Thue-Morse words



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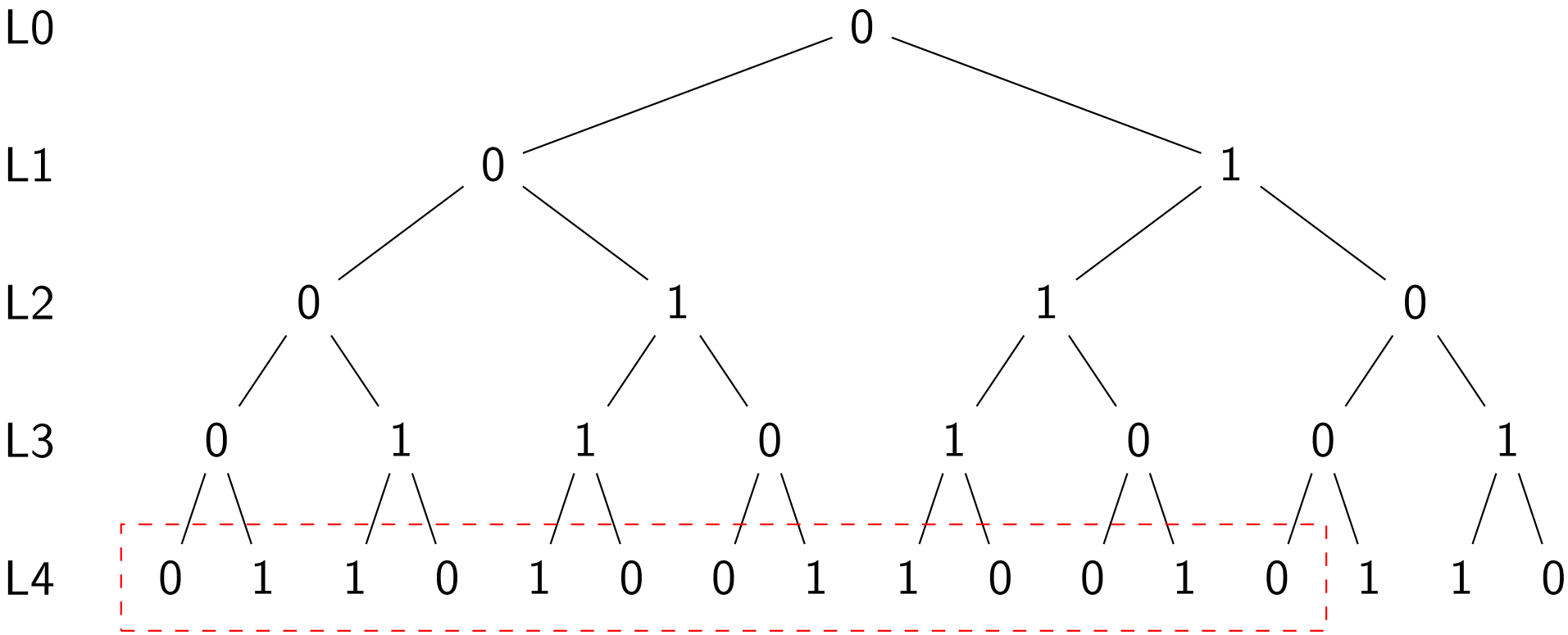

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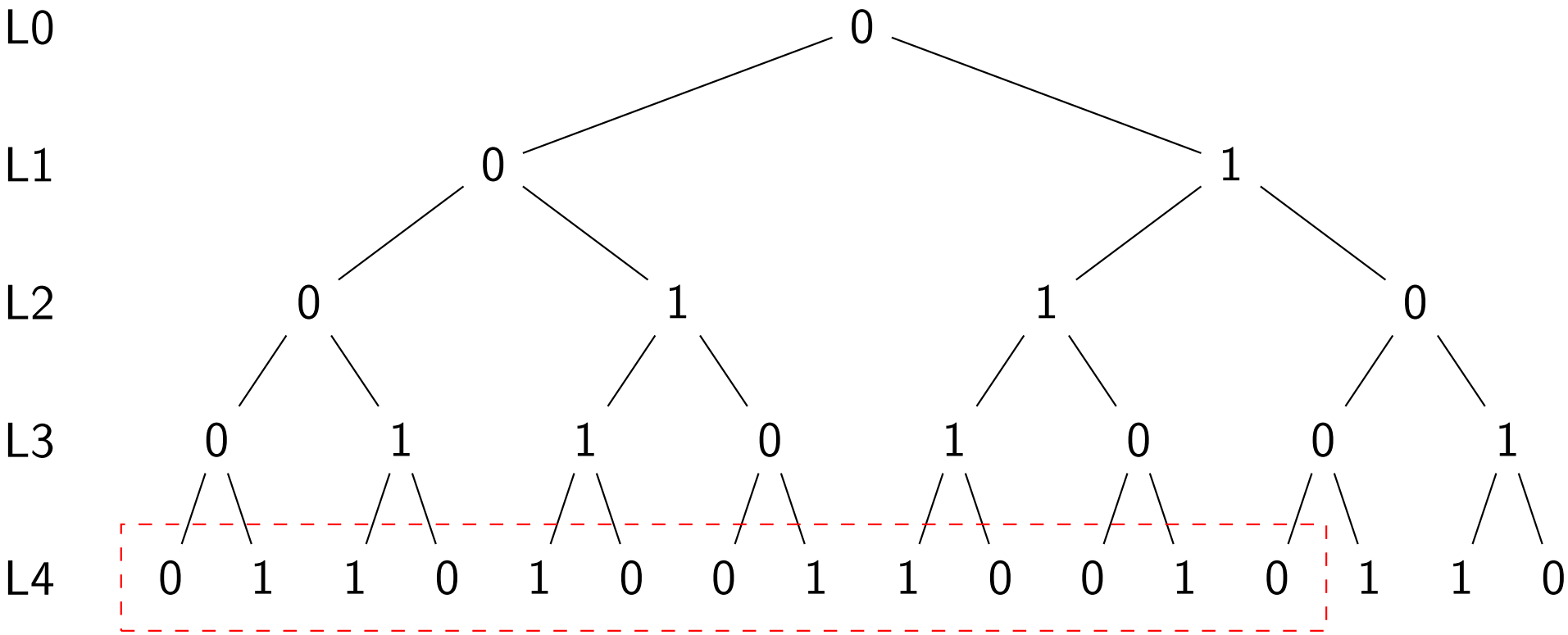


$$V = \{0, 1\}, R = \{0 \to 01, 1 \to 10\}, S = 0$$
$$d = 4, n = 13, \tau = \{0 \to 0, 1 \to 1\}$$

# L-systems - Results

**Theorem 2**

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## Proof.

The L-system $L_d = (V, R, S, \tau, d, n)$ where $V = \{0, 1\}$, $S = 0$, $R(0) = 001$, $R(1) = 1$, $\tau = id$, and $n = 2^{d+1} - 1$, trivially has constant size as $d$ (and $n$) grows. The first strings of the family generated by this system by growing $d$, are 0, 001, 0010011, 001001100100111, and so on.
For $d \geq 4$, in $exp(L_d)$ there exist $d^2/8 + d/4$ distinct substrings of length $d$, and $\delta(exp(L_d)) \geq (d^2/8 + d/4)/d = d/8 + 1/4$.
Hence, on the family $\{exp(L_d) \mid d \geq 0\}$, it holds that $\delta = \Omega(\ell \log n)$. □

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**Theorem 3**

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**Proof.**

Kociumaka et al. showed a family of strings with $\delta = O(1)$ that needs $\Omega(\log n) = \Omega(\delta \log n)$ space, to be represented with any method. On the other hand, an L-system of size $\ell$ is described in $O(\ell)$ space. Therefore $\ell = \Omega(\delta \log n)$ on this family. □

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**Proof.**

Consider the smallest grammar $G = (V, \Sigma, R, S)$ of size $g$ and height $h$ generating the string $w[1, n]$. We can easily construct an L-system of size $O(g)$ simulating $G$ by defining rules $a \to a$ for the terminals, keeping the rules for the variables of the grammar, letting $R(S)$ be the axiom, and $d = h$. □

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**Theorem 5**

*For any L-system $L = (V, R, S, \tau, d, n)$ of size $\ell$ generating $w[1, n]$, there is a context-free grammar of size $(d + 1)\ell$ generating $w$. If the morphism represented by $R$ is expanding, then the grammar is of size $O(\ell \log n)$.*

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**Proof.**

We create a variant of each terminal sub indicated by the level where it is used in the parse tree. This allows us control the level where to stop but multiplies the size by the expansion height. □

# L-systems - Results

**Theorem 6**

*Let $w \in \Sigma^*$, $\phi : \Sigma^* \to \Sigma^*$ be a non-erasing automorphism, and let $\tau : \Sigma \to \Sigma$ be a coding. Then $\ell = \Theta(1)$ on the family $\{\tau(\phi^d(w)) \mid d > 0\}$.*

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**Proof.**

We can easily simulate the automorphism $\phi$ on the L-system $L = (\Sigma, R, w, \tau, d, n)$ of fixed size, with $R(a) = \phi(a)$ and $n = |\tau(\phi^d(w))|$. This system generates $\tau(\phi^d(w))$, and as $d$ grows, it does not change its size. □

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On the other hand, it is probably NP-hard to compute $\ell$.

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A *NU-system* is a tuple $N = (V, R, S, \tau, d, n)$, which works on the same way that an L-system, but with the addition of *extractions rules*, that is, $R : V \to (V \cup E)^+$ with $E = \{A(l)[i, j] \mid A \in V, l, i, j \in \mathbb{N}\}$.

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- $\nu(w)$ is the size of the smallest NU-system for $w$.

# NU-systems - Example

Consider a NU-system with rules $a \to a$, $b \to b$, $A \to ab$, and $S \to A\ S(2)[1,4]$, $d = 2$, $n = 6$, and $\tau = id$. The derivation is then generated as follows:

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L_0 & = & S & \longrightarrow \quad A\ S(2)[1]\ S(2)[2]\ S(2)[3]\ S(2)[4] \\
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 & & & \quad\ A\ a\ b\ (A(1)[1])(1)[1]\ (A(1)[2])(1)[1] \\
L_1 & = & A\ a\ b\ a\ b & \longleftarrow \quad A\ a\ b\ a(1)[1]\ b(1)[1] \\
L_2 & = & a\ b\ a\ b\ a\ b. &
\end{array}
$$

The number of extraction symbols is bounded by $|V| \cdot max(l) \cdot n$, so it is computable if the derivation of a NU-system eventually ends.

# NU-systems - Results

**Theorem 7**

*It always holds that* $\nu = O(\min(\ell, m))$.

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**Proof.**

It is trivial that $\nu \leq \ell$ because L-systems are a particular case of NU-systems.

With respect to $m$, we do the same that we did for simulating grammars with L-systems. We handle the extractions rules of the form $A \to B[i, j]$ by replacing them with rules $A \to B(h)[i, j]$, where $h$ is the height of the macro system. □

# NU-systems - Results

**Theorem 8**

*Let $N_1$ and $N_2$ be NU-systems generating $w_1$ and $w_2$, respectively. Then there are NU-systems of size $O(size(N_1) + size(N_2))$ that generate $w_1 \cdot w_2$ and the* composition *of $N_1 \circ N_2$ ($N_2$ with axiom $w_1$).*

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**Proof.**

We can use the the NU-systems $N_1$ and $N_2$ to construct new NU-systems satisfying these claims. □

# NU-systems - Summary

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On the other hand $\nu$ is probably NP-hard and does not have a guaranteed polynomial decompression time.

# Contents

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- Can we find a good theoretical lower bound for self-similarity schemes?

# Closing

Thank you very much for your attention.