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ANNEX 1

THE CONTINUITIES OF THE POINT-TO-SET MAPS, DEFINITIONS AND EQUIVALENCES

0. Introduction

In the theory of set-valued mapping, two kinds of continuity have been developed. For each of them, very closely related definitions have been given using on one hand (Hill (1927), Kuratowsky and Hahn (1932), Bouligand and Blanc (1933)) ordering inclusion properties in terms of limit of sequences of sets and on the other hand (Hahn (1932), Choquet (1948), Berge (1959)) topological properties of the "inverse image". The connexions between these definitions are given in the following. All throughout the paper, we consider a map F from X into $\mathcal{P}(Y)$, the set of subsets of Y, where X, Y are Hausdorff spaces. Particular assumptions (for example, first countability) will be specified when necessary; it is to be noticed that none of the assumptions that are given, can be deleted. The properties are presented without proofs, the majority of the results being stated in the literature (for complete proofs and counter examples see [2]).

0.1. Notation

$\mathcal{V}(x)$ the family of the neighbourhoods of $x \in X$,
$N' \subset \mathbf{N}$ will always denote an infinite subset of $\mathbf{N}$,
$\{x_n\}_{\mathbf{N}}$ a sequence of points in X ($\{x_n\}_{N'}$ an extracted subsequence)
$[A] \leftrightarrows^H [B]$ means:

$$\begin{cases} A \Rightarrow B, \text{ and} \\ B \Rightarrow A \text{ if assumption } H \text{ is verified.} \end{cases}$$

(for the meaning of such a specified assumption, see the footnotes to Diagrams 1 and 2).

1. Limits of sets (Hahn and Kuratowski)

Let $\{A_n\}_{\mathbf{N}}$ be a given sequence of subsets of a topological space Y.

1.1. Lower limit of $\{A_n\}_{\mathbf{N}}$

$\underline{\lim}_{\mathbf{N}} A_n$ denotes the lower limit of the sequence $\{A_n\}_{\mathbf{N}}$, i.e. the subset of Y (possibly empty) that consists of points x satisfying

$$(\forall V \in \mathcal{V}(x))(\exists n_0) n \geq n_0 \Rightarrow A_n \cap V \neq \emptyset,$$

1.2. Upper limit of $\{A_n\}$

$\overline{\lim}_{\mathrm{N}} A_n$ denotes the upper limit of the sequence $\{A_n\}_{\mathrm{N}}$, i.e. the subset of Y (possibly empty) that consists of all points x satisfying

$$(\forall V \in \mathcal{V}(x))(\forall n)(\exists n' \geq n) A_{n'} \cap V \neq \emptyset.$$

1.3. Properties

(a) $\underline{\lim}_{\mathrm{N}} A_n \subset \overline{\lim}_{\mathrm{N}} A_n$ and these subsets are closed in Y.
(b) $x \in \underline{\lim}_{\mathrm{N}} A_n \leftrightarrows_{H_1} (\exists \{x_n\}_{\mathrm{N}} \to x)(\exists n_0)(\forall n \geq n_0) x_n \in A_n$.
(c) $x \in \overline{\lim}_{\mathrm{N}} A_n \leftrightarrows_{H_1} (\exists N' \subset \mathrm{N})(\exists \{x_n\}_{N'} \to x)(\forall n \in N') x_n \in A_n$.
(d) If $\mathcal{O}$ is opened in Y, then

$$(\exists N' \subset \mathrm{N})(\forall n \in N') A_n \cap \mathcal{O} = \emptyset \to (\underline{\lim}_{\mathrm{N}} A_n) \cap \mathcal{O} = \emptyset.$$

(e) If G is compact (or G sequentially compact)

$$(\overline{\lim}_{\mathrm{N}} A_n) \cap G = \emptyset \Rightarrow (\exists n_0)(\forall n \geq n_0) A_n \cap G = \emptyset.$$

(f) If Y is a metric space with compact closed-balls, G closed, then

$$\left.\begin{array}{l} A_n \cap G \neq \emptyset \quad (\forall n) \\ A_n \text{ connected subset of } Y \quad (\forall n) \\ \overline{\lim}_{\mathrm{N}} A_n \neq \emptyset \text{ and compact} \end{array}\right\} \Rightarrow (\overline{\lim}_{\mathrm{N}} A_n) \cap G \neq \emptyset.$$

2. First kind of continuity: the lower continuity

2.1

We present four definitions that have been introduced in the literature.

Definition 2.1 (Hill, Kuratowski, Hahn, Blanc). The map F is said to be *lower semi continuous by inclusion* (L.S.C.) at $x \in X$ if and only if

$$(\forall \{x_n\}_{\mathrm{N}} \to x) F(x) \subset \underline{\lim}_{\mathrm{N}} F(x_n).$$

Definition 2.2 (Hahn, Choquet, Berge). The map F is said to be *lower semi continuous* (l.s.c.), at $\bar{x} \in X$ if and only if

$$(\forall \mathcal{O} \subset Y, \text{opened}) \; F(x) \cap \mathcal{O} \neq \emptyset \Rightarrow (\exists V \in \mathcal{V}(x))(\forall x' \in V) F(x') \cap \mathcal{O} \neq \emptyset.$$

Definition 2.3 (Debreu, Hogan, Huard). The map F is said to be *opened* (or *lower continuous*) at $x \in X$ if and only if

$$\left.\begin{array}{l} (\forall \{x_n \in X\}_{\mathrm{N}} \to x) \\ (\forall y \in F(x)) \end{array}\right\} (\exists \{y_n \in Y\}_{\mathrm{N}} \to y)(\exists n_0)(\forall n \geq n_0) y_n \in F(x_n).$$

Definition 2.4 (Brisac). The map F is said to be *lower semi continuous*, (l.s.c.) at $x \in X$ if and only if

$$\overline{F(x)} = \{y \in Y \mid (\forall V \in \mathcal{V}(y))\{x' \in X \mid F(x') \cap V \neq \emptyset\} \in \mathcal{V}(x)\}$$

where $\overline{F(x)}$ denotes the closure of $F(x)$.

2.2. *Equivalence properties*

Diagram 1 shows the connections between these definitions; the reader is referred to its footnote for the meaning of hypothesis H_i $(i = 1, 2)$. These equivalences are given for the definitions at $x \in X$.

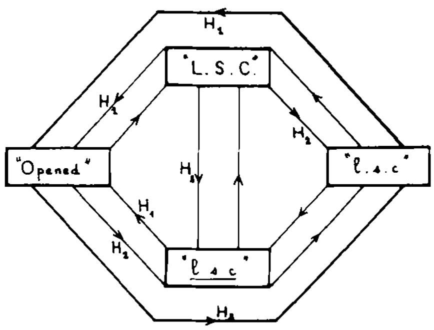

Diagram 1.
H_1: Y satisfies the "first axiom of countability";
H_2: "there exists at $x \in X$ a countable base of neighbourhoods".

Remark 1. The notion of lower-continuity is extended to the whole space X by assuming the l.s.c. at every $x \in X$.

A characterization of the lower semi continuity (l.s.c.) on X is given by the following relation (Berge, Choquet): F is l.s.c. on X if and only if

$$\{x \in X \mid F(x) \cap \mathcal{O} \neq \emptyset\} \text{ is opened in } X \text{ for every opened } \mathcal{O} \text{ in } Y.$$

Remark 2. It is worth noticing that all the previous definitions are equivalent if X and Y are first countable Hausdorff spaces. This is the case for metric spaces.

3. Second kind of continuity: the upper continuity

3.1

Definition 3.1 (Hill, Kuratowski, Hahn, Bouligand, Choquet). The map F is said to be *upper semi continuous by inclusion*, (U.S.C.) at $x \in X$ if and only if

$$(\forall \{x_n \in X\}_N \to x)\ \overline{\lim}_N F(x_n) \subset F(x).$$

Definition 3.2 (Hahn, Choquet, Berge). The map F is said to be *upper semi continuous* (u.s.c.) at $x \in X$ if and only if

$$(\forall \mathcal{O} \subset Y, \text{ opened})\ F(x) \subset \mathcal{O} \Rightarrow (\exists V \in \mathcal{V}(x))x' \in V \Rightarrow F(x') \subset \mathcal{O}.$$

Definition 3.3 (Debreu, Hogan, Huard). The map F is said to be *closed* (or *upper continuous*) at $x \in X$ if and only if

$$\left.\begin{array}{l}(\forall \{x_n \in X\}_{\mathbb{N}} \to x) \\ (\forall \{y_n \in Y\}_{\mathbb{N}} \to y \text{ such that } y_n \in F(x_n)) \quad (\forall n)\end{array}\right\} \Rightarrow y \in F(x).$$

Definition 3.4 (Choquet). The map F is said to be *weakly upper semi continuous* (w.u.s.c.) at $x \in X$ if and only if

$$(\forall y \notin F(x))(\exists U \in \mathcal{V}(x))(\exists V \in \mathcal{V}(y))x' \in U \Rightarrow F(x') \cap V = \emptyset.$$

Remark 3. Choquet calls "strong upper semi continuity" the property of Definition 3.2.

Remark 4. Both Definitions 3.1 and 3.4 imply "$F(x)$ closed", but Definition 3.3 only implies "$F(x)$ sequentially closed".

3.2. Equivalence properties

Diagram 2 shows the connections between these definitions (the meaning of assumptions H_i ($i = 1, 4$) is given by its footnote). Beside these equivalences, the following proposition gives a relation between U.S.C. and u.s.c. in a particular interesting context.

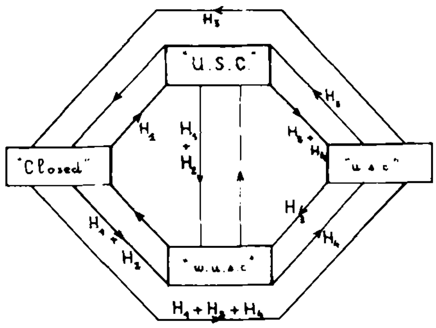

Diagram 2.

These following equivalences are given for the definitions at $x \in X$.

H_1: Y satisfies the "first axiom of countability";

H_2: "there exists at $x \in X$ a countable base of neighbourhoods";

H_3: Y is a regular space and $F(x)$ is closed;

H_4: $\overline{Y - F(x)}$ is compact (in particular, if Y is compact, H_4 is fulfilled).

Proposition 3. *If Y is a metric space with compact closed balls, then*

$$\left.\begin{array}{l} F(x)\neq\emptyset \text{ and compact} \\ (\exists C \text{ compact})(\exists V\in\mathcal{V}(x))x'\in V\Rightarrow\left\{\begin{array}{l}F(x') \text{ connected} \\ F(x')\cap C\neq\emptyset\end{array}\right.\end{array}\right\}\Rightarrow[\underset{\text{at } x}{\text{USC}}\Rightarrow\underset{\text{at } x}{\text{u.s.c}}].$$

Remark 5. It is to be noticed that Definitions 3.1, 3.3 and 3.4 are equivalent if the spaces X and Y are first countable Hausdorff spaces (in particular, if X, Y are metric spaces).

Remark 6. It is worth noticing that Berge defines the u.s.c. of a map over the whole space X by the two following conditions:

$$\begin{cases} F \text{ is u.s.c. at every } x\in X, \\ F \text{ is compact-valued.}\end{cases}$$

Bibliography

(1) The reader is referred to the general reference list of this Study (more precisely, papers with a mark in column 2).

(2) J.P. Delahaye, J. Denel, "Equivalences des continuités des applications multivoques dans des espaces topologiques", Publication n° 111, Laboratoire de Calcul, Université de Lille (1978).

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