

Algorithmic complexity for short binary strings applied to psychology: a primer

Nicolas Gauvrit · Hector Zenil · Jean-Paul Delahaye ·
Fernando Soler-Toscano

© Psychonomic Society, Inc. 2013

Abstract As human randomness production has come to be more closely studied and used to assess executive functions (especially inhibition), many normative measures for assessing the degree to which a sequence is randomlike have been suggested. However, each of these measures focuses on one feature of randomness, leading researchers to have to use multiple measures. Although algorithmic complexity has been suggested as a means for overcoming this inconvenience, it has never been used, because standard Kolmogorov complexity is inapplicable to short strings (e.g., of length $l \leq 50$), due to both computational and theoretical limitations. Here, we describe a novel technique (the *coding theorem method*) based on the calculation of a universal distribution, which yields an objective and universal measure of algorithmic complexity for short strings that approximates Kolmogorov–Chaitin complexity.

Keywords Algorithmic complexity · Randomness · Subjective probability

The production of randomness by humans requires high-level cognitive abilities, such as sustained attention and inhibition,

N. Gauvrit (✉)
CHART (PARIS-reasoning), University of Paris VIII and EPHE,
Paris, France
e-mail: ngauvrit@me.com

H. Zenil
Unit of Computational Medicine, Center for Molecular Medicine,
Karolinska Institute, Stockholm, Sweden

J.-P. Delahaye
Laboratoire d'Informatique Fondamentale de Lille, Lille, France

F. Soler-Toscano
Grupo de Lógica, Lenguaje e Información, Universidad de Sevilla,
Seville, Spain

and is impaired by poor working memory. Unlike other frontal neuropsychological tests, random generation tasks possess specific features of interest: Their demand on executive functions, especially inhibition processes, is high, and more important, training does not reduce this demand through automatization (Towse & Cheshire, 2007). On the contrary, generating a randomlike sequence requires continuous avoidance of any routine, thus preempting any automatized success.

Random generation tasks have been widely used in the last few decades to assess working memory, especially (sustained) inhibitive abilities (Miyake et al., 2000), in normal subjects as well as in patients suffering from a wide variety of pathologies. In normal subjects, random generation varies with personal characteristics or states, such as belief in the paranormal (Brugger, Landis, & Regard, 1990) or cultural background (Streng, Lesmana, & Suryani, 2009; Vandewiele, D'Hondt, Didillon, Iwawaki, & Mwamwendat, 1986). It affords insight into the cognitive effects of aging (Heuer, Janczyk, & Kunde, 2010), hemispheric neglect (Loetscher & Brugger, 2008), schizophrenia (Chan, Hui, Chiu, Chan, & Lam, 2011), aphasia (Proios, Asaridou, & Brugger, 2008), and Down syndrome (Rinehart, Bradshaw, Moss, Brereton, & Tonge, 2006).

As a rule, random generation tasks involve generating a randomlike sequence of digits (Loetscher, Bockisch, & Brugger, 2009), nouns (Heuer et al., 2010), words (Taylor, Salmon, Monsch, & Brugger, 2005), or heads-or-tails (Hahn & Warren, 2009). Some authors have also offered a choice of more neutral items, such as dots, (e.g., in the classical Mittenecker test; Mittenecker, 1958). Formally, however, these cases all amount to producing sequences of bits—that is, 0 or 1 digits—since any object can be coded in this way. In most studies, the sequence length lies between 5 and 50 items. Measuring the formal “randomness” of a given short sequence (say, of length 5 to 50) is thus a crucial challenge. Apart from any objective and formal definition of randomness, researchers regularly use a variety of indices, none of which is

sufficient by itself because of the profound limitations they all exhibit. Recently, for instance, Schulter, Mittenecker, and Papousek (2010) provided software calculating the most widely used of such measures applied to the case of the Mittenecker Pointing Test, together with a comprehensive overview of the usual coefficients of randomness in behavioral and cognitive research. These tools provide a new way to describe how a given sequence differs from a truly random one. However, it is not fully satisfactory: Multiple unsatisfactory measures do not result in a satisfactory description.

The usual measures of randomness

The most common coefficients used to assess the quality of a pseudorandom production may be classified into three large varieties according to their main focus.

Departure from uniformity

The simplest coefficients—even though they may rely upon sophisticated theories—are based on the mere distribution of outcomes and are, therefore, independent of the order of the outcomes. In brief, they amount to the calculation of a distance between the observed distribution and the theoretical flat distribution, just as a chi-square would do.

Information theory (Shannon & Weaver, 1949) is often used as a basis for assessing randomness. Given a finite sequence s of N symbols repeatedly chosen from a set of n elements, the average symbol information, also called entropy, is given by $H(s) = -\sum_i p_i \log_2(p_i)$, where p_i is the relative frequency of an item i in the finite sequence. This entropy is maximal when the relative frequencies are all equal, when it amounts to $H_{max} = \log_2(n)$.

Symbol redundancy (SR; see Schulter et al., 2010) is an example of a coefficient arising from information theory. It is defined to be $SR = 1 - H/H_{max}$, where H is the entropy. SR is no more than a measure of departure from uniformity. A sequence's SR does not depend on all aspects of the sequence, but only on the relative frequencies of each item it comprises. According to SR, a sequence made up of 0s and 1s, such as 000001, is weakly random, as expected, but 010101, 000111, 100101 are exactly equivalent to each other, since they all minimize SR to 0. This is the most obvious limitation of SR as a global measure of randomness, as well as of any measure relying on the mere distribution of symbols.

Normality

Beyond values depending on the distribution of individual symbols, one may consider pairs (dyads) or sequences of three (triads) adjacent outcomes. In a truly (infinite) random sequence, any dyad should appear with the same probability.

Any distance between the distribution of dyads or triads (and so on) and uniformity may therefore be thought of as a measure of randomness. This is precisely what context redundancy CR_1 and coefficient of constraint CC_1 assess (Schulter et al., 2010).

One may also consider dyads of outcomes separated by 1, 2, or k elements in the sequence, which is done, for instance, through CC_k and CR_k coefficients, a generalization of CC_1 and CR_1 . Here, we group methods of this kind under the rubric *normality assessment*, for a reason that will soon become clear.

In mathematics, a sequence of digits d_1, \dots, d_n, \dots , with d_n lying within $[0, b]$ —or equivalently, the real number written $0.d_1d_2d_3\dots$ in base $(b+1)$ —is said to be normal in base b if the asymptotic distribution of any dyad, triad, or finite sequences of k consecutive digits is uniform. Any such series also manifests what may seem a stronger property: Dyads of outcomes separated by a certain fixed number of other outcomes show, in the long run, a uniform distribution. Therefore, a normal sequence will be considered random by any normality assessment method.

Eventually, there will exist sequences produced by simple rules that are normal. The Copeland–Erdős sequence, arising from the concatenation of prime numbers (235711131719 . . .), is an example. The Champernowne sequence (Champernowne, 1933), a concatenation of all integers from one on (123456789101112131415 . . . ; or in base 2, 11011100101110111 . . .), is an even more simple example. There even exist rules to generate absolutely normal sequences—that is, numbers that are normal in any base b (Becher & Figueira, 2002). Thus, sequences exist that would pass any normality test of randomness, despite being produced by a simple rule—which contravenes the notion that randomness is defined by an absence of regularity.

Gaps

Another variety of randomness coefficient is worked out using the rank distances between two identical items. For instance, in the sequence 12311, the distances between occurrences of the symbol 1 are 3 and 1. The frequency distribution of repetition distances (gaps) and the median of repetition gap distribution are based on the study of the distance between two identical outcomes. They have proved useful in detecting the so-called cycling bias: People tend to repeat an item only after they have used every other available item once (Ginsburg & Karpiuk, 1994).

Gap methods are as limited as normality assessment is: A normal sequence will pass these tests and be considered truly random, even if a naive rule produces it.

Cognitive complexity

Apart from these attempts, a variety of measures to assess *cognitive* complexity or randomness were put forward during

the 1960s and 1970s (for an overview, see Simon, 1972). These calibrations of cognitive complexity are built for the most part on the idea of algorithmic complexity, although in practice the link with Turing machines (TMs; the TM is at the root of Kolmogorov complexity theory) is evanescent. Usually, they consist of a series of rules or pattern descriptions that human subjects supposedly use and are aware of. The complexity of a string is defined as a function of the minimum number of rules one has to use to produce the string in question. These indices depend heavily on the rules and pattern descriptions chosen and lack mathematical rationale but are psychologically sound. Their purpose is not to set a normative and formal measure of randomness through complexity but to capture the nature of human complexity perception.

Very recently, a more sophisticated approach based on changes has been suggested (Aksentijevic & Gibson, 2012). Given an n -long string s , define the change function of s , $c(s)$, as the $(n-1)$ -long string $f(s)$ whose i th term is 0 if $s(i) = s(i + 1)$, and 1 if not. From a string s of length n , build an $n \times n$ matrix whose j th line L_j is $c(L_{j-1})$. Complexity is defined as a weighted sum of the matrix coefficients. Aksentijevic and Gibson claimed that this *change complexity* is an alternative to the unfortunately uncomputable algorithmic complexity.

However, there is no mathematical rationale for using change complexity as a normative measure of complexity or randomness. Rather than being a computable version of algorithmic complexity, it seems like a refinement of the usual normality assessment tools. It captures some local structural aspects of the string to be rated and will consider every normal sequence as perfectly random, even though it is produced by a simple algorithm. Change complexity is also heavily reliant on psychological considerations and subjective choices. For these reasons, it should be classified with the previous attempts to capture *perceived* randomness, but not as a formal measure of complexity. However useful it may be in capturing human perceived complexity or the production of pseudorandom strings, it is a bad candidate for rating human pseudorandom productions in a normative and objective fashion.

Psychological justifications and limitations

Notwithstanding their potential limitations, all the above-mentioned coefficients have proved useful in detecting some common biases in random generation. SR-like values capture outcome biases—the overuse of certain symbols (Nickerson, 2002). Normality assessment accurately spots alternation biases—the tendency to avoid using the same symbol twice (e.g., HH or TT)—or the inverse *repetition bias*. Context redundancy has also been linked with cognitive flexibility (Stoffers, Berendse, Deijen, & Wolters, 2001), of which it constitutes an estimate. Gaps and related methods would diagnose the cycling bias, a tendency to repeat the same pattern or to exhaust every available symbol before a

repetition (Ginsburg & Karpiuk, 1994). For instance, if the available symbols are 1, 2, 3, 4, a subject might choose 1, 3, 4, 2, 1, postponing the second appearance of “1” until after every symbol has occurred once. Repetition avoidance is known to affect outcomes as far as six ranks forward (Chapanis, 1995), a bias that gap methods shed light on.

It is unclear whether these measures happen to capture the basic biases in human random generation or whether, unfortunately, researchers have focused on these biases simply because they have had tools at their disposal for diagnosing them. As we have seen in the previous sections, a normal sequence such as that suggested by Champernowne (1933), which is highly nonrandom to the extent that it is generated by a simplistic rule, would meet all random criteria using symbol distribution, normality assessment, and gap methods, as well as change complexity.

At this point, we may list three senses in which the usual randomness estimates are unsatisfactory. First, they do not capture nonstandard biases in randomness, such as the existence of a simple generation rule, when such a rule begins to produce sequences bearing some resemblance to a truly random one—for example, a normal sequence. Only a few features of a random sequence are captured by these tailored measures. Second, they lack a theoretical basis. Despite being based on formal probabilistic properties, they are nevertheless not subsumed by a theory of randomness. In fact, they neither use nor provide any definition of a random sequence. Third, partly as an upshot of the first two points, several coefficients are needed to sketch an acceptable diagnosis of the quality of randomness, whereas a single measure would allow the comparison of sequences.

In what follows, we describe a method for overcoming the uncomputability of algorithmic complexity and, thus, arrive at a measure for short strings that possesses interesting features. First, it captures the complexity of a string in a single numerical value. This allows comparison between any two sequences, even if they are not of equal length. For instance, developmental studies of randomness (see below for a preliminary experiment) will benefit from having a global estimation of the quality of randomness. Second, because it is based on the Kolmogorov theory of randomness, it will detect any deviation from randomness or computable regularity, whatever the source of the discrepancy. And third, because it is linked with algorithmic probability (the probability that a string is produced by a random program), it serves as a bridge between complexity and the Bayesian approach. Knowing the algorithmic complexity of a string s , one is able to compute the probability that it is produced by a random process, a result that would be meaningful within the new paradigm of subjective probability.

We must, however, underline the fact that the algorithmic complexity of short strings alone will not yield a description of any discrepancy detected between a theoretical and an experimental string. Investigating the manner in which human

pseudorandom productions differ from truly random sequences (which is, of course, of great importance for psychology) will remain the domain of more specific measures focusing on particular characteristics of interest, such as the indices mentioned above.

Algorithmic complexity

The need for a universal approach that does not focus on specific arbitrary features, as well as for a theoretical framework defining finite randomness, has been expressed by psychologists (Barbasz, Stettner, Wierzchon, Piotrowski, & Barbasz, 2008) and addressed outside psychology by the mathematicians Andrei Kolmogorov (1965) and Gregory Chaitin (1966). The theory of algorithmic complexity, also known as Kolmogorov–Chaitin complexity (Li & Vitányi, 2008), provides a formal definition of a random sequence. In this section, we first provide an overview of this theory, identify its limits, and then suggest an approach for overcoming them, following recent developments in the field (Delahaye & Zenil, 2007). This approach is consistent with current theories of randomness perception in psychology. Falk and Konold (1997) have suggested that our subjective assessment of randomness relies on a perception of some sort of complexity, thus linking perceived randomness and complexity. However, they do not provide any practical means of measuring randomness for short strings based on complexity.

Our goal is to provide a new conceptual and practical tool enabling the assessment of randomness through algorithmic complexity for short strings within an objective and universal approach. It is thus meant not to capture human representations of randomness, but to verify that pseudorandom human behaviors can be rated on a numerical scale bridging randomness and complexity.

Formal languages and automata theory

When considering the complexity of an object, one may think of said object as simple if it can be described in a few words. One can, for example, describe a string of a million alternating zeros and ones (01010101 . . .) as “a million times 01” and say that the string is simple given its short description. However, it is fair to point out that the description of something is highly dependent on the choice of language. The strings a language can compress depend on the language used, since any string (even a random-looking one) can be encoded using a one-word-long description by mapping it onto any word of a suitable language. A language can always be tailor-made to describe any given object by using something that describes the object in a very simple way. Due to these difficulties, it was not until the arrival of the theory of computation, and the precise definition of a computing machine by Alan Turing

(1936), that the theory of algorithmic information found a formal framework on which it could build a definition of complexity and randomness.

Today, the TM model represents the basic framework underlying most concepts in computer science, including the definition of algorithmic randomness. A TM is an abstract model of a digital computer formalizing and defining the idea of mechanical calculation.

A TM consists of a list of rules capable of manipulating a contiguous list of cells (usually pictured as a tape with an unlimited quantity of cells in both directions) and an access pointer (an active cell) equipped with a reading head. The TM can be in any one of a finite set of states Q , numbered from 1 to n , with 1 being the state at which the machine starts a computation. There is a distinct $n+1$ state, called the *halting* state, at which the machine halts. Each tape cell can contain a 0 or a 1 (sometimes there is a special blank symbol filling the tape). Time is discrete, and the instants of time (steps) are ordered from 0, 1, . . ., with 0 being the time at which the machine starts its computation. At any given time, the head is positioned over a particular cell. The head can move right or left, reading the tape. At time 0, the head is situated over a particular cell on the tape called the *initial* cell, and the finite program starts in state 1. At time 0, the content of the tape is called the *machine input*. Turing’s seminal contribution was the demonstration that there exist TMs (today named *universal* TMs, which we will denote simply by UTM) capable of simulating any other TM (Turing, 1936). One does not need specific computers to perform specific tasks; a single programmable computer could perform any conceivable task (we are now capable of running a word processor on the same digital computer on which we play chess games). He also proved that programs and data do not have any particular feature that distinguishes them from one another, given that a program can always be the input of another TM, and data can always be embedded in a program. A full description of a TM can be written in a 5-tuples notation as follows: $\{s_i, k_i, s_{i+1}, k_{i+1}, d\}$, where s_i is the scanned symbol under the head, k_i the state at time t , s_{i+1} the symbol to write at time $t+1$, k_{i+1} the state to be in at time $t+1$, and d the head movement either to the right or to the left. A TM can perform the following, which in combination define a single operation: (1) Write an element from $A = \{0,1\}$, (2) shift the head one cell to the left or right, and (3) change to state $k \in Q$.

When the machine is running, it executes one such operation at a time (one every step) until it reaches the halting state—if it ever does. At the end of a computation, the TM will have produced an output described by the contiguous cells of the tape over which the head passed before halting. A TM may or may not halt, and if it does not halt, it is considered to have produced no output. Turing also showed that there is no procedure to determine whether a TM will halt or not (Turing, 1936). This is set forth as the *undecidability of the*

halting problem, identified with the common term *uncomputability*.

A definition of algorithmic complexity

The basic idea at the root of algorithmic complexity is that a string is random (or complex) if it cannot be produced by a program much shorter in length than the string itself. The algorithmic complexity $C(s)$ of a bit string s is formally defined as the length in bits of the shortest program p that outputs it when running on a UTM U . Formally, $C(s) = \min_p \{ |p| : U(p) = s \}$, where p is a program and $U(p)$ the corresponding output (Chaitin, 1966; Kolmogorov, 1965).

One is compelled to use a UTM because one wants a machine that is capable of printing out any possible string s for which one may want to calculate $C(s)$. Nevertheless, no computable procedure exists to calculate $C(s)$, due to Turing's undecidability of the halting problem, so $C(s)$ is usually approximated through the use of lossless compression algorithms, such as the Lempel–Ziv algorithm (Lempel & Ziv, 1976). The length of the compressed string s is actually an upper bound of $C(s)$ —hence, a sufficient test of nonrandomness. However, lossless compression algorithms do not help when strings are short—shorter than, for example, the compression program length in bits. For instance, usual lossless compression algorithms give complexity above 40 for strings of length 1 or 2. Moreover, strange phenomena happen, such as the string “11111” being rated as far less complex than “111” or “111111.” This and other arguments against the use of compression algorithms as estimates for complexity are detailed in a recent paper by Soler-Toscano, Zenil, Delahaye, and Gauvrit (2012, section 3.1 and Fig. 1). In practice, a string of length $l \leq 50$ cannot be compressed by lossless compression algorithms because the compressed file should include the decompression algorithm (otherwise, one can compress anything to an arbitrary symbol with no further instructions), which usually needs more than 40 bits. Hence, Kolmogorov complexity by itself is of no use in the situations encountered by psychologists. This is probably the reason why, although algorithmic complexity has been successfully used in biology to assess the complexity of DNA sequences, it has, to the best of our knowledge, never been directly used in experimental psychology.

The choice of Turing machine matters

The definition of algorithmic complexity clearly seems to depend on the specific UTM U , and one may ask whether there exists a different UTM yielding different values for $C(s)$. The following theorem indicates that the definition of algorithmic complexity makes sense even if measured on different UTMs (or if desired, using different programming languages):

Theorem 1 (invariance). If U and U' are two UTMs and $C_U(s)$ and $C_{U'}(s)$ the algorithmic complexity of a binary string s using U and U' , there exists a constant c that does not depend on s , such that for all binary strings, $|C_U(s) - C_{U'}(s)| < c$.

The proof of this theorem is quite straightforward (see, e.g., Calude, 2002). The ability of universal machines to efficiently simulate each other implies that there is a program p_1 for the universal machine U that allows U to simulate U' . One can think of p_1 as a translator in U' for U (the length of p_1 is an upper bound for c). Let p_2 be the shortest program producing s according to U' . Then there is a program for U made of p_1 and p_2 and generating s as an output.

For strings from a certain length on, this theorem indicates that one will asymptotically approach the same complexity value regardless of the choice of UTM. However, constants can make the calculation of $C(s)$ for *short* strings profoundly dependent on the UTM used (Delahaye & Zenil, 2008). Both the problem of noncomputability and the problems posed by short strings may account for the absence of applications in psychology and other social sciences.

Algorithmic probability

Solomonoff (1960) had the idea of describing the likelihood of a UTM generating a sequence with a randomly generated input program, having in mind a program to solve the problem of induction. Further formalizing this concept, Levin defined the *algorithmic probability* of a string s as the probability that a random program (the bits of which are produced with a fair coin flip) would produce s when running on a UTM U (Zvonkin & Levin, 1970). Formally, Levin's probability measure is defined as $m(s) = \sum_{\{p: U(p)=s\}} 1/2^{|p|}$.

If $m(s)$ as a probability measure is not to be greater than 1, U has, however, to be what is called a prefix-free TM—that is, a machine that only accepts programs that are not the beginning of any other program (i.e., there exists an “END” sequence finishing any acceptable program). Levin's probability measure induces a distribution over programs producing s , assigning a greater chance that the shortest program is the one actually generating s .

The coding theorem method

The *coding theorem* connects algorithmic probability (or frequency of production) to algorithmic complexity (Calude, 2002):

Theorem 2 (coding theorem). For any finite string s and prefix-free universal Turing machine, U , $|- \log_2(m(s)) - C_U(s)| \leq M$, for a fixed number M independent of s .

The coding theorem offers an alternative for approximating $C(s)$, given that one can evaluate $C_U(s)$ by calculating $\log_2(m(s))$. This idea—the use of the coding theorem as an alternative to compression algorithms as a means of estimating $C(s)$ —has already been suggested by Delahaye and Zenil (2007). To this end, they sampled the space of all TMs up to a certain size (two symbols and four states) from which a frequency distribution of the outputs generated by these machines was produced. We denote by $D(n)$ the probability distribution of all two-symbol n -state TMs. In other words, $D(n)$ provides the production frequency of a string s , taking into account all two-symbol n -state TMs. A string is counted in $D(n)$ if it is the output of a TM—that is, the content of the contiguous cells on the machine tape that the head has gone through before halting.

Solving the halting problem for small machines

$D(n)$, like $m(s)$ and $C(s)$, is a semicomputable function. One may use the results from a problem popular among computer scientists, called the *busy beaver problem*, to calculate $D(n)$ as an approximation of $m(s)$.

The busy beaver problem is the problem of finding a value $S(n)$ for the maximum runtime of a TM with n states before halting, when starting from an empty input. Radó (1962) proved that finding $S(n)$ for any n is impossible given the undecidability of the halting problem, but for $n < 5$ state and two-symbol TMs, $S(n)$ is known. These values are arrived at by essentially simulating all different n -state machines and proving that the remaining nonhalting machines will never halt (which is possible only because the machines are small and simple). Values of $S(n)$ for $n < 5$ are known (Brady, 1983; Shen & Radó, 1965). These values of the busy beaver problem allowed Delahaye and Zenil (2008, 2012) to numerically approximate the complexity of strings using the coding theorem.¹ In what follows, we will apply a similar procedure to the set of all two-symbol five-state TMs.

D(5) and K_5

To date, $D(4)$ has been the most advanced distribution released. This article represents a significant improvement in both accuracy and scope, providing new data from all two-symbol five-state TMs, denoted by $D(5)$. The following is the description of the procedure that we call the *coding theorem method*.

Procedure

A TM with n states using the binary alphabet $\{0,1\}$ requires a set of $2n$ instructions, one for each possible $(state, symbol)$

¹ The tables produced by Zenil and Delahaye are available online at the following URL: www.algorithmicnature.org.

pair. Each of these instructions provides a new state, a symbol to write on the tape, and a direction in which to move the machine head.

Figure 1a is a graphical representation of an example of a machine with three states. States are represented by black drops pointing in different directions (see legend). The halting state is represented by a black dot. The 6 two-cell pictures represent the instructions of the TM. In each instruction, the top cell represents the state and the symbol read before applying the instruction. For example, the first instruction is applied when the TM is in state 1 reading symbol “0”. The bottom part of each instruction represents the symbol written, the new state, and the direction (the new state is drawn to the right or left of the cell that is written on). So we can read the first instruction thus: When the machine is in state 1 reading the symbol “0”, it replaces “0” with “1”, moves to the left and changes to state 3. Note that when an instruction moves to the halting state (in Fig. 1a, it happens only when the machine is in state 2 reading “0”), the head does not move, halting in the same cell. Figure 1b displays the execution of this TM (on an unbounded tape). At the beginning, all the cells are blank. When the machine reads a blank, it identifies it as a 0. When the TM halts, the tape reads “101”, which is therefore the string produced by this particular machine.

Each one of the $2n$ instructions in a TM with n states can be one of the $4n + 2$ different combinations of $(new_state, new_symbol, direction)$. When new_state is the halting state, there are only two possible instructions (writing “0” or “1”). In other cases, there are $4n$ possibilities (n possible values for new_state , 2 for new_symbol , and 2 for $direction$). That adds up to $4n + 2$. Then, considering the $2n$ instructions of a TM with n states, we obtain the result that the number of different machines with n states is $(4n + 2)^{2n}$.

Our experiment involved obtaining the output of all these machines for $n = 5$ —that is, 26 559 922 791 424 different machines—machines starting on a blank tape filled with “0,”

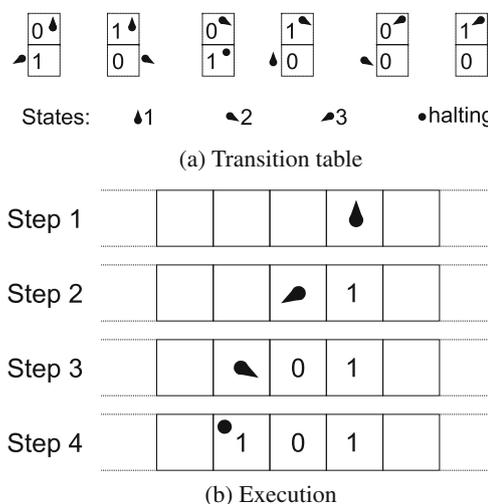


Fig. 1 An example of a machine with three states

as well as machines starting on a blank tape filled with “1” (in order to consider both symbols as possible blank symbols).

In fact, we did not run all the machines because we exploited some symmetries to considerably reduce the number of machines to run. For example, there is a left–right symmetry that allowed us to run only machines with the starting instruction moving in a fixed direction—for example, to the right. Then the output of some machines starting on the left is the reverse string of that produced by the left–right symmetric machines. There is also a “0”–“1” symmetry that made it unnecessary to run every machine twice, one for each possible blank symbol. We set “0” as the blank symbol, and for every string produced, we also considered its “0”–“1” symmetric string (interchanging “0” and “1”). Also, we did not run many trivial machines that we knew in advance would not halt (machines with the initial transition remaining in the initial state) or that would halt in just one step (machines with the initial transition to the halting state). These reductions allowed us to run only $2(n-1)(4n+2)^{2n-1}$, which for $n=5$ meant taking up 4/11 of the original space. At any rate, it took 18 days in the cluster at the CICA (Centro Informático Científico de Andalucía) supercomputing service, using a C++ simulator. Having obtained the output string of the machines that were run, we applied the proper completions to obtain the full set of strings representing the complete space of the different $(4n+2)^{2n}$ machines running on both “0”-filled and “1”-filled tapes.

This computation is very demanding, but needs to be run only once, which is now done.

Setting the runtime

The maximum number of steps that a TM with five states and two symbols can run before halting is still unknown. There are known machines running 47,176,870 steps, but we could not set that runtime for all the machines, due to time limitations. So we decided to set a maximum runtime of 500 and keep only the output strings produced by machines requiring that number of steps at most to halt.

As an experiment to try and ascertain how many machines we were leaving out, we ran 1.23×10^{10} random TMs for up to 5,000 steps. Figure 2 gives the number of TMs that halt after exactly n steps. The rapid decrease denotes the fact that very few machines actually halt after more than 200 steps. Among these, only 50 machines halted after 500 steps and before 5,000. Thus, according to this experiment, our election of a runtime of 500 steps leaves out only an insignificant portion of the machines.

Not all machines were run for 500 steps, since we set some filters to detect nonhalting machines before consuming the

maximum runtime. For example, cycles of period two are easy to detect. They are produced when in steps s and $s+2$, the tape is identical and the machine is in the same state and the same position. When this is the case, the cycle will be repeated infinitely.

Calculating $D(5)$

The output of our experiment is the list of strings produced by machines with five states and two symbols, together with the number of occurrences of each string. We proceeded to construct the function $D(5)(s)$, which returns the probability that a halting machine with five states and two symbols produces string s when running on a blank tape (without specifying the blank symbol). Formally,

$$D(5)(s) = \frac{\text{number of occurrences of } s}{\text{number of halting executions}}$$

Note that the number of halting executions is equal to the sum of occurrences of all the strings produced, so the sum of $D(5)$ for all strings is 1.

Features of K_5

Function $D(5)(s)$ is a natural approximation of $m(s)$, from which an approximation of Kolmogorov–Chaitin complexity may be derived by use of the coding theorem. Formally, we define the $D(5)$ -based K_5 algorithmic complexity for short strings by the formula

$$K_5(s) = -\log_2(D(5)(s)).$$

Every string of length $l \leq 11$ was a five-state two-symbol TM output, as were 4,094 of the 4,096 strings of length 12 and some strings up to length 49, making for a total of 99,608 strings.

Examples of top and bottom strings

Table 1 displays the 12 most frequent and, therefore, least complex strings of length 12 according to $D(5)$, as well as the top 12 least frequent (hence, most complex) strings. It confirms our expectations: Strings with apparent regularity (“000000000000” or “010101010101”) are simple, and the more complex ones exhibit no such visibly regular pattern.

Detection of global regularities

Some binary sequences may seem simple from a global viewpoint because they show symmetry (1011 1101) or repetition (1011 1011). Let us consider the string $s = 1011$ as an example. We have $D(5)(s) = 3.267414 \times 10^{-3}$. The repetitions $ss = 10111011$ have a much lower probability,

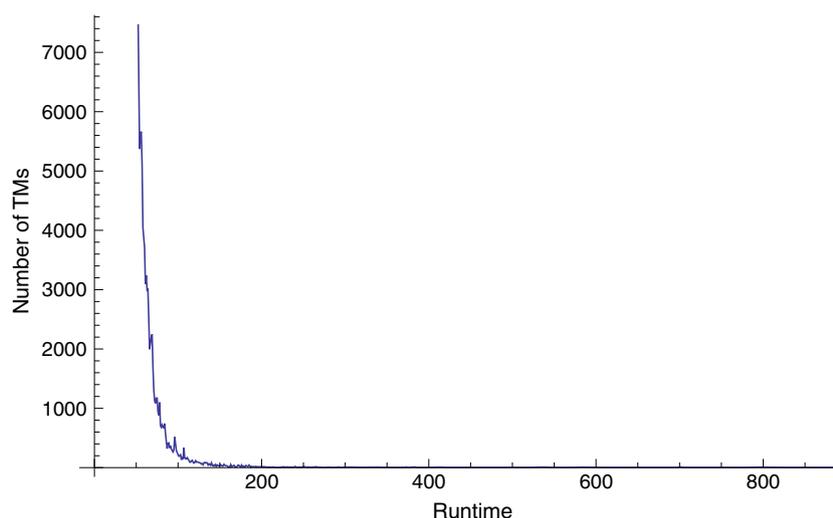


Fig. 2 Runtime distribution in $D(5)$

$D(5)(ss) = 4.645999 \times 10^{-7}$. This is not surprising considering the fact that ss is much longer than s , but we may then wish to consider other strings based on s . In what follows, we will consider three methods (repetition, symmetrization, 0-complementation). The repetition of s is $ss = 10111011$, the “symmetrized” $s\bar{s} = 10111101$, and the 0-complementation 10110000 . These three strings of the same length have different probabilities (4.645999×10^{-7} , 5.335785×10^{-7} , and 3.649934×10^{-7} , respectively).

Let us now consider all strings of length 3 to 6 and their symmetrizations, 0-complementations, and repetitions. Figure 3 provides a visual representation of the results. In each case, even the minimum mean between the means of symmetrized, complemented, and repeated patterns (dotted horizontal line) lies in the upper tail of the $D(5)$ distribution for $2n$ -length strings, and this is even more obvious with longer strings. Symmetry, complementation, and repetition are, on average, recognized by $D(5)$.

Alternations

Human pseudorandom productions have been described as exhibiting too much alternation: When trying to

Table 1 Top 12 simple strings according to $D(5)$ (top of the table) and 12 more complex strings (bottom)

111111111111	000000000000	101010101010
010101010101	111111111110	100000000000
011111111111	000000000001	110101010101
101010101011	010101010100	001010101010
011000111001	000100110111	111000110100
110100111000	001011000111	000111001011
110100011100	110001110100	001110001011
001011100011	110000111100	001111000011

behave randomly, humans have a proclivity to produce “1” after “0” or “0” after “1.” The mean frequency of alternation in a random binary string is .5, but slightly superior frequencies (typically around .6) have been reported in human pseudorandom generation (e.g., Tubau & López-Moliner, 2009).

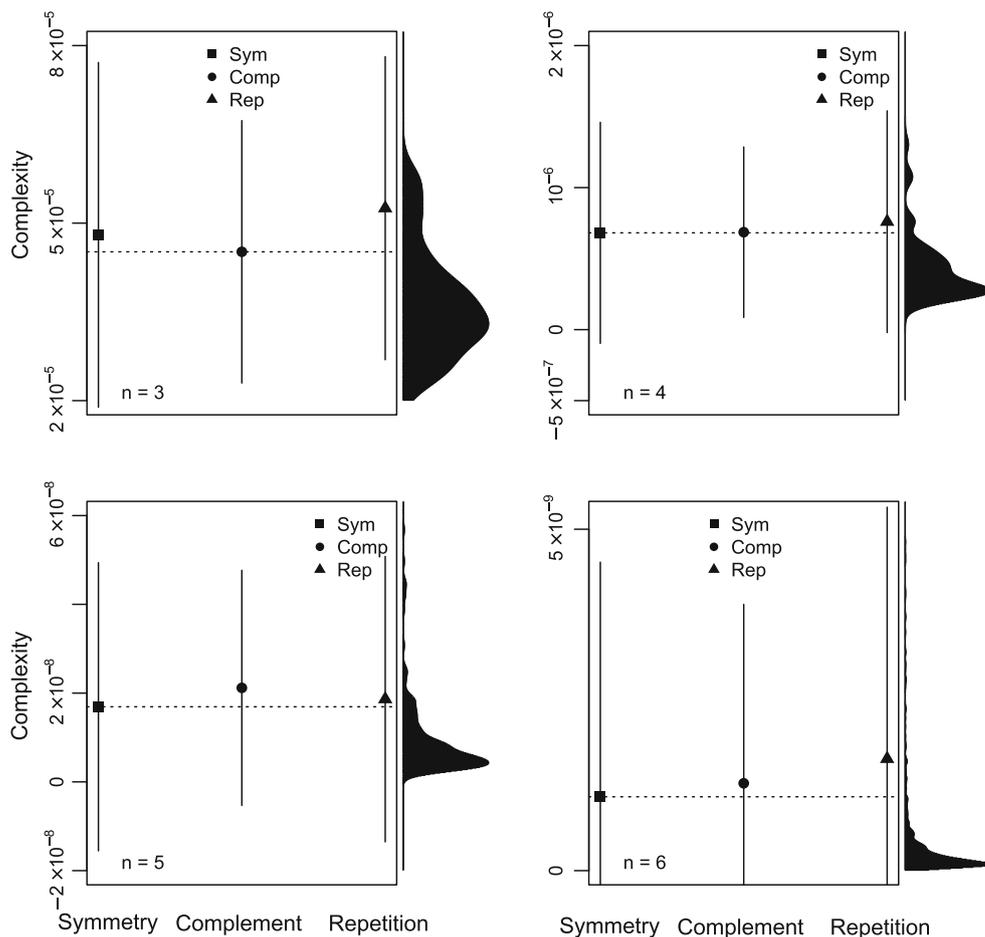
It is now widely believed that when human subjects try to behave randomly, they actually try to maximize the complexity of their responses. However, due to cognitive limitations, we probably are unable to produce binary sequences of maximal complexity, because this would require a too complicated algorithm. This theoretical intuition recently received experimental support when researchers found that children were more attracted by mildly complex patterns (Kidd, Piantadosi, & Aslin, 2012).

Figure 4 shows that binary strings of medium or mildly high complexity tend to exhibit an excess of alternations, whereas both simple and very complex strings tend to have an excess of repetitions. Complexity is a means to a deeper understanding of alternation bias. Formally, we suggest the following conjecture: When human subjects try to behave randomly and produce a binary string, they try to generate a complex sequence. Because their efforts are only partially rewarded, they produce strings of mild or just-above medium complexity, which usually have too much alternation.

Bayesian use of $D(5)$

Given a string s , we may now compute the probability of it being truly random (event R) against the hypothesis that it has been created by a TM (event M). Let l be a length (for instance, $l=12$). In this paragraph, we will denote the conditional probability of $D(5)$ when the length is l by P_l . Set the

Fig. 3 Mean \pm standard deviation of $D(5)$ of $2n$ -long strings given by processes of symmetrization (Sym), 0-complementation (Comp), and repetition (Rep) of all n -long strings. The dotted horizontal line shows the minimum mean among Sym, Comp, and Rep. The density of $D(5)$ for all $2n$ -long strings is given in the right margin



prior probability that the underlying process is random to $P(R)=1/2$. We then have $P(s|R)=2^{-l}$ and

$$P(R|s) = \frac{P(s|R)P(R)}{P(s)},$$

with

$$P(s) = P(s|M)P(M) + P(s|R)P(R) = \frac{P_l(s)}{2} + 2^{-(l+1)}.$$

From this we derive

$$P(R|s) = \frac{1}{1 + 2^l P_l(s)}.$$

Table 2 gives a few examples of random binary strings of length 12, together with their probability of being random and the number of 1s they include. As we can see from this table and as Fig. 5 visually confirms, the more complex strings (which also have a great probability of being random) are likely to be balanced, with approximately six 1s. This again may help us understand the so-called “belief in the law of small numbers” (Tversky & Kahneman, 1971), according to

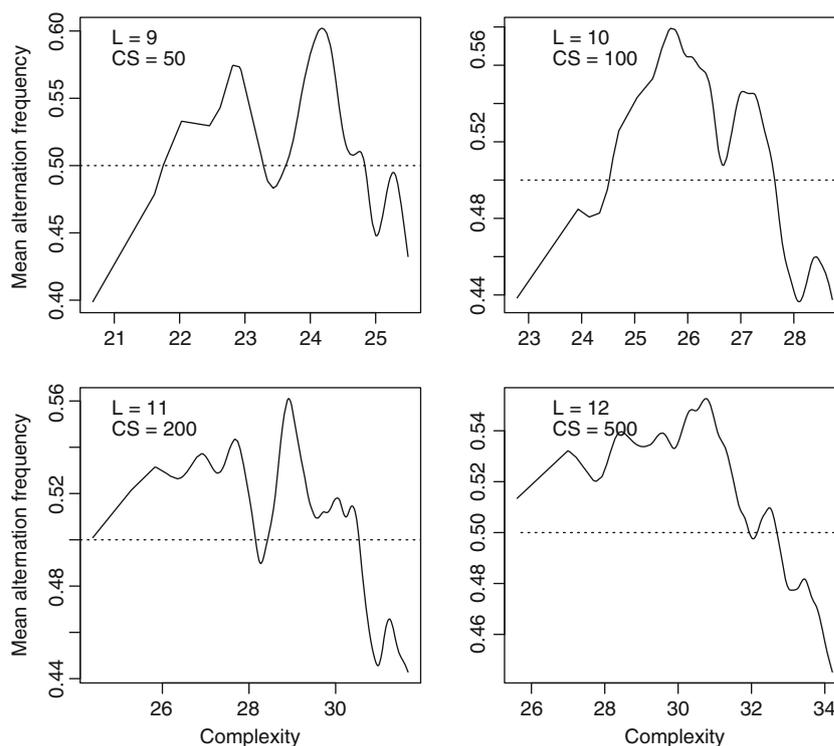
which subjects wrongly tend to generate equal numbers of each alternative while trying to produce random binary strings. Once again, this could be a result of an attempt to generate complex responses—that is, strings that are more likely to be random.

Material and tools

An Online Algorithmic Complexity Calculator (OACC) implementing the technique presented herein to approximate the Kolmogorov complexity of short strings and meant for the use of the research community is accessible at <http://www.complexitycalculator.com>. The online calculator provides a means for approximating the complexity of binary short strings for which no other method has existed hitherto by taking advantage of the formal connections among these measures and putting together several concepts and results from theoretical computer science.

The calculator is a long-term project to develop an online tool implementing the semicomputable measures of complexity described in this article and is expected to be expanded in the future. It currently implements numerical approximations of Kolmogorov complexity and algorithmic probability for

Fig. 4 For each length $l=9$ to 12, we select clusters of CS strings and compute the mean frequency of alternations within these strings. The diagram displays the smooth-spline curve of the resulting function. The number CS of strings is chosen according to the total number of strings of length l , to ensure readability. The clusters are built thus: Strings are sorted by increasing complexity. For a given complexity k , let j be the rank of the first string with complexity above k . Then the corresponding cluster comprises all strings with rank j to $j+CS-1$



short binary strings following the numerical methods described herein, strings where lossless compression algorithms fail when used to approximate their Kolmogorov complexity. Hence, it provides a complementary and alternative method to compression algorithms.

The OACC is intended to provide a comprehensive framework of universal mathematical measures of randomness, structure, and simplicity for researchers and professionals. It can be used to provide objective measures of complexity in a very wide range of disciplines, from bioinformatics to psychometrics, from linguistics to finance. More measures, more data, and better

approximations will be gradually incorporated in the future, covering a wider range of objects, such as longer binary strings, nonbinary strings, and n -dimensional arrays (such as images).

The raw data set with the full calculation of $D(5)$ can also be downloaded from <http://complexitycalculator.com/D5.csv> in Comma Separated Values format. For R-users, a script is also available. It defines a function returning the complexity of any binary string up to length 11 and many strings of length 12 to 49. It also computes the algorithmic probability $D(5)$ and the probability of being random in a Bayesian approach $P(R|s)$. This R-script may be used to compute complexity in data frames, thus allowing a statistical exploration of complexity.

Table 2 A few examples of 12-long strings s with the associated probability of being random $P(R|s)$ and the number of 1s (6 meaning that s is perfectly balanced)

s	$P(R s)$	Ones
110100011100	.975	6
101100100011	.951	6
110001111001	.942	7
101100111100	.933	7
100000000011	.519	3
110101001011	.515	7
101110110111	.180	9
101111101111	.151	10
010101010101	.097	6
010000000000	.079	1
000000000000	.017	0

A comparison with entropy

Let us now briefly compare algorithmic complexity as defined above and the most widespread alternative measure of randomness: entropy. We will focus on binary strings of length 12. There are $2^{12} = 4,096$ different binary strings of length 12.

Among the 4,096 binary strings of length 12, entropy distinguishes six groups. It is not defined (or equals 0) for strings “000000000000” and “111111111111”, and for any other string, it depends only on the number of the least frequent digit, which can appear 1, 2, 3, 4, 5, or 6 times.

Algorithmic complexity is far more fine-grained, distinguishing 371 different levels of complexity.

The class of maximum entropy ($H = -\log_2(2) = 1$) includes 461 different strings, of varying algorithmic complexity. Within this group, we find strings of very low complexity, such as

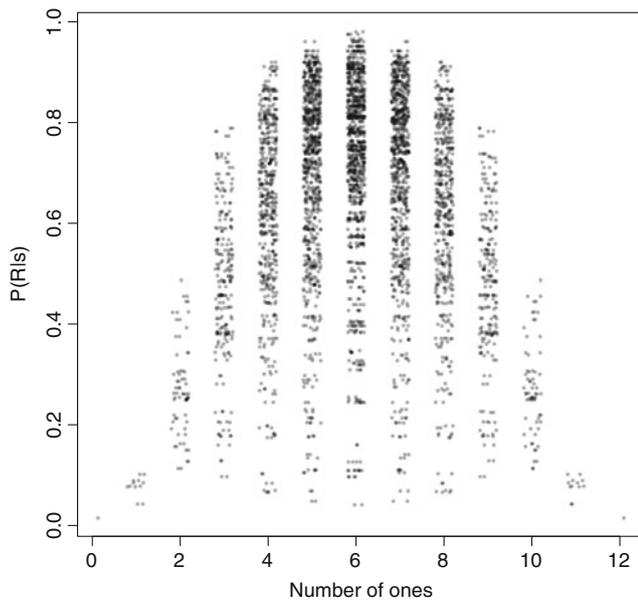


Fig. 5 Scatterplot of the probability of being random against the number of 1s in all 12-long strings. A jitter method has been applied on the number of 1s to increase readability

“010101010101” ($K_5 = 26.99$), and of high complexity, such as “100101110001” ($K_5 = 36.06$).

Some strings of lower entropy sometimes have greater K_5 than do others with higher entropy. For instance, the string “110011011111” has an entropy of 0.811, fourth among six possible levels of entropy, but a complexity of $K_5 = 33.36$. Thus, “110011011111” is less random than “010101010101” according to its entropy but more random according to its K_5 . All the strings of very low entropy ($H < 0.5$) also have low complexity, but both simple and complex strings can have medium or high entropy.

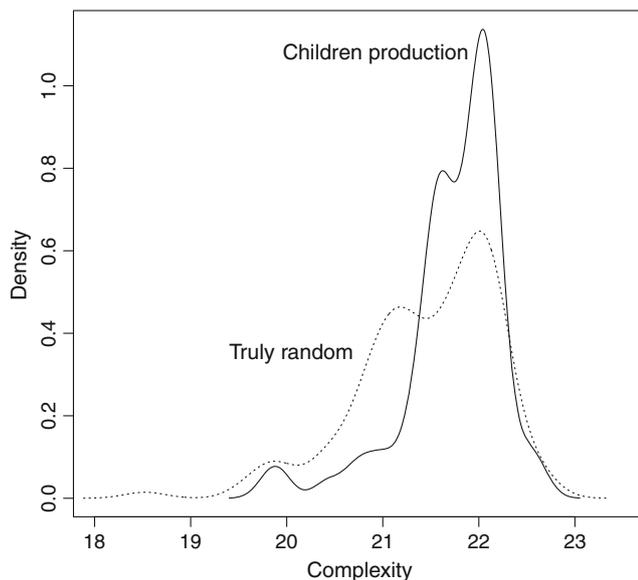


Fig. 6 Density of truly random strings’ complexity (dotted) and of children’s productions, as approximated by Gaussian kernel

On a more theoretical level, the Kolmogorov definition of a random string is equivalent to another mathematical construct, *effective statistical tests*: A string is random if and only if it passes every effective statistical test. An important mathematical advantage of the theory of algorithmic complexity is that it constitutes the only tool that would detect any computable regularity (Martin-Löf, 1966). The measures mentioned above (entropy, contextual redundancy, change complexity, gaps, etc.) may be thought of as particular computable statistical tests of randomness. They each aim at detecting one variety of nonrandomness. On the other hand, the Kolmogorov approach is equivalent to running every computable statistical test at the same time. In a way, it can be thought of as a universal statistical test of randomness.

A case study

In this short section, we present and briefly discuss experimental data gathered via an algorithmic-complexity-for-short-strings approach, for illustrative purposes. This study is part of a larger experiment still in progress that investigates the evolution of subjective probability with age in children and teenagers.

Subjects and procedure

Sixty-eight children (35 male, 33 female) 7–11 years of age participated in this experiment (mean age \pm SD: 8.7 ± 1.1). All the children were pupils in five different classes from two public schools in France, grades 2–4. The youngest subjects (second graders) were the first to be tested, followed 4 months later by the third graders, and 4 months after that by the fourth graders.

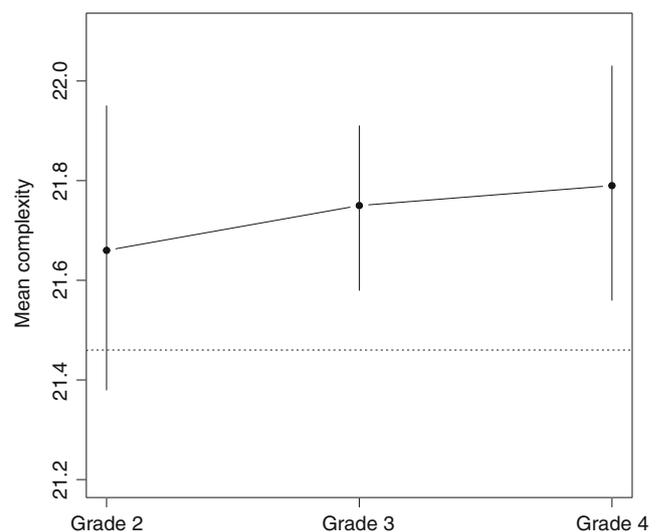


Fig. 7 Mean complexity K_5 and 95 % confidence intervals by grade. The dotted line shows the mean complexity of truly random binary strings of length 8

Each child was received individually in a room in the school, during class time. A token was presented to him or her. It was green on one side and red on the other. The experimenter explained what was meant by a “toss,” and the token was then hidden so it would not distract the subject.

The instruction given to the child was to imagine that s/he tossed the token 8 times and to say out loud which side, green or red, appeared each time. Each child thus produces one binary string that is 8 units long.

Global complexity

The complexity of all possible strings of length 8 runs from 18.53 to 22.68, with a median of 21.60. The more frequent strings generated by the children are 00101101, 01001101, and 11010010, appearing 4 times each. These strings have medial complexities (21.58, 21.37, and 21.58, respectively).

On the whole, however, children do better than a true random process in terms of complexity. The mean complexity of a random string of length 8 is 21.46, whereas children’s productions have a mean complexity of 21.74 ($SD: 0.52$). This is significantly more than 21.46, $t(67) = 4.35$, $p < 5 \times 10^{-5}$. Of course, choosing a random TM instead of a random string would lead to an overrepresentation of simple sequences; hence, children’s productions are also “better” than random TM outputs.

Figure 6 shows the density of complexities of truly random strings and of those produced by children. It strongly suggests that the main difference between truly random strings (i.e., every string shares the same probability of being picked up) and human production is that humans contrive to avoid very simple strings. They do not generate more high-complexity sequences than would be expected from chance, but they do produce a lot of mildly complex strings.

Effect of education

The period from 7 to 11 years is known to be one of relative stability. Piaget and Inhelder (1951) claimed that the notion of probability could not be grasped before the formal stage, at around age 11. In an experimental study of randomness perception in 7- to 16-year-olds, Green (1986) finds no evolution between 7 and 11.

One aim of the present experiment was to ascertain whether there is an evolution in subjective randomness at this particularly stable stage. Concerning complexity, we expected either no differences from grade 2 through grade 4 or an increase in complexity. Figure 7 displays the means and confidence intervals by grades. A one-way ANOVA was applied to the data. Although there is no significant difference, $F(2, 65) = 0.68$, $p = .41$, a small effect (Cohen, 1992) ($\eta^2 = .11$) appears in our sample and shows a slight progression of complexity with grade.

The main finding of this experiment is that even young children 7–11 years of age, from grade 2 to 4, can do better than picking a string from a uniform distribution to build up strings that “look random.” This behavior is not an irrational bias: Any string is certainly as likely to appear by chance, but not all of them show the same amount of evidence for a random underlying process. The possible slight evolution of complexity from grades 2 to 4 confirms that this period of childhood is relatively stable.

Conclusion

For decades, researchers have relied upon algorithmic Kolmogorov–Chaitin complexity as a means to rate complexity and randomness in an objective way. Algorithmic complexity may seem to be a purely theoretical apparatus, useful only in abstract operations on infinite sequences. It has turned out to be useful in practice when applied to DNA sequences or when long strings (of millions of digits, for instance) are involved. In such cases, compression methods seem to overcome the impossibility of an exact calculation of algorithmic complexity. Compression methods unfortunately do not apply to short strings, which we encounter in the behavioral sciences. Since amending the theory to make it suitable for short strings was believed to be impossible, researchers have followed two routes:

1. Some renounced the ideal of a normative unique measure of complexity and used a variety of indices focusing on special features of the strings they had to measure.
2. Others switched from universal TMs to tailor-made cognitive “machines” designed according to what they thought was important for humans. In doing so, they generated good estimates of *perceived* complexity but lost the universality of algorithmic complexity.

The coding theorem method described above allows us to use algorithmic complexity even for very short strings. As we have shown, this may shed new light on some biases, such as the alternation bias or the belief in the law of small numbers: They may be simple consequences of a tendency to be mildly complex. Because K_5 measures an objective complexity, we cannot expect human productions to be as complex as the more complex possible strings, although a mild and above-average complexity may be expected.

As Peter Brugger puts it,² until now “a complexity measure for short strings [was] badly needed.”

² Personal communication, November 6, 2012.

References

- Aksentijevic, A., & Gibson, K. (2012). Complexity equals change. *Cognitive Systems Research*, 15–16, 1–16.
- Barbasz, J., Stettner, Z., Wierzchon, M., Piotrowski, K., & Barbasz, A. (2008). How to estimate randomness in random sequence generation tasks? *Polish Psychological Bulletin*, 39(1), 42–46.
- Becher, V., & Figueira, S. (2002). An example of a computable absolutely normal number. *Theoretical Computer Science*, 270, 947–958.
- Brady, A. (1983). The determination of the value of Rado's noncomputable function $\Sigma(k)$ for four-state Turing machines. *Mathematics of Computation*, 40(162), 647–665.
- Brugger, P., Landis, T., & Regard, M. (1990). A "sheep-goat effect" in repetition avoidance: Extra-sensory perception as an effect of subjective probability? *British Journal of Psychology*, 81, 455–468.
- Calude, C. (2002). *Information and randomness. an algorithmic perspective* (2nd, revised and extended éd.). Springer-Verlag.
- Chaitin, G. (1966). On the length of programs for computing finite binary sequences. *Journal of the ACM*, 13(4), 547–569.
- Champernowne, D. G. (1933). The construction of decimals normal in the scale of ten. *Journal of the London Mathematical Society*, 8, 254–260.
- Chan, K., Hui, C., Chiu, C., Chan, S., & Lam, M. (2011). Random number generation deficit in early schizophrenia. *Perceptual and Motor Skills*, 112, 91–103.
- Chapanis, A. (1995). Human production of "random" numbers. *Perceptual and Motor Skills*, 81, 1347–1363.
- Cohen, J. (1992). A power primer. *Psychological Bulletin*, 112(1), 155–159.
- Delahaye, J.-P., & Zenil, H. (2007). Randomness and complexity. from Leibnitz to Chaitin. In (p 123-130). World Scientific Publishing.
- Delahaye, J.-P., & Zenil, H. (2008). Towards a stable definition of Kolmogorov-Chaitin complexity. (Retrieved from arXiv: 0804.3459v3 [cs.IT])
- Delahaye, J.-P., & Zenil, H. (2012). Numerical evaluation of the complexity of short strings: A glance into the innermost structure of algorithmic randomness. *Applied Mathematics and Computation*, 219, 63–77.
- Falk, R., & Konold, C. (1997). Making sense of randomness: Implicit encoding as a basis for judgment. *Psychological Review*, 104(2), 301–318.
- Ginsburg, N., & Karpiuk, P. (1994). Random generation: Analysis of the responses. *Perceptual and Motor Skills*, 79, 1059–1067.
- Green, D. (1986). Children's understanding of randomness: Report of a survey of 1600 children aged 7-11 years. In *Proceedings of icots 2* (p. 287-291).
- Hahn, U., & Warren, P. (2009). Perceptions of randomness: Why three heads are better than four. *Psychological Review*, 116(2), 454–461.
- Heuer, H., Janczyk, M., & Kunde, W. (2010). Random noun generation in younger and older adults. *The Quarterly Journal of Experimental Psychology*, 63(3), 465–478.
- Kidd, C., Piantadosi, S., & Aslin, R. (2012). The goldilocks effect: Human infants allocate attention to visual sequences that are neither too simple nor too complex. *PLoS*, 7(1), e36399.
- Kolmogorov, A. (1965). Three approaches to the quantitative definition of information. *Problems of Information and Transmission*, 1(1), 1–7.
- Lempel, A., & Ziv, J. (1976). On the complexity of finite sequences. *IEEE Transactions in Information Theory*, 22(1), 75–81.
- Li, M., & Vitányi, P. (2008). *An introduction to Kolmogorov complexity and its applications*. Springer Verlag.
- Loetscher, T., Bockisch, C., & Brugger, P. (2009). Eye position predicts what number you have in mind. *Current Biology*, 20(6), 264–265.
- Loetscher, T., & Brugger, P. (2008). Random number generation in neglect patients reveals enhanced response stereotypy, but no neglect in number space. *Neuropsychologia*, 47, 276–279.
- Martin-Löf, P. (1966). The definition of random sequences. *Information and Control*, 9, 602–619.
- Mittenecker, E. (1958). Die analyse zufälliger reaktionsfolgen [the analysis of "random" action sequences]. *Zeitschrift für Experimentelle und Angewandte Psychologie*, 5, 45–60.
- Miyake, A., Friedman, N. P., Emerson, M. J., Witzki, A. H., Howerter, A., & Wager, T. (2000). The unity and diversity of executive functions and their contributions to frontal lobe tasks: A latent variable analysis. *Cognitive Psychology*, 41, 49–100.
- Nickerson, R. (2002). The production and perception of randomness. *Psychological Review*, 109(2), 330–357.
- Piaget, J., & Inhelder, B. (1951). *La genèse de l'idée de hasard chez l'enfant*. Presses Universitaires de France.
- Proios, H., Asaridou, S. S., & Brugger, P. (2008). Random number generation in patients with aphasia: A test of executive functions. *Acta Neuropsychologica*, 6, 157–168.
- Radó, T. (1962). On non-computable functions. *Bell System Technical Journal*, 41(3), 877–884.
- Rinehart, N. J., Bradshaw, J. L., Moss, S. A., Brereton, A. V., & Tonge, B. J. (2006). Pseudo-random number generation in children with high-functioning autism and Asperger's disorder. *Autism*, 10(1), 70–85.
- Schulter, G., Mittenecker, E., & Papousek, I. (2010). A computer program for testing and analyzing random generation behavior in normal and clinical samples: The Mittenecker pointing test. *Behavior Research Methods*, 42, 333–341.
- Shannon, C. E., & Weaver, W. (1949). *The mathematical theory of communication*. University of Illinois Press.
- Shen, L., & Radó, T. (1965). Computer studies of turing machine problems. *Journal of the ACM*, 12(2), 196–212.
- Simon, H. A. (1972). Complexity and the representation of patterned sequences of symbols. *Psychological Review*, 79, 369–382.
- Soler-Toscano, F., Zenil, H., Delahaye, J.-P., & Gauvrit, N. (2012). Calculating Kolmogorov complexity from the output frequency distributions of small turing machines. (Retrieved from arXiv: 1211.1302v1 [cs.IT])
- Solomonoff, R. (1960). *A preliminary report on a general theory of inductive inference (Rapport technique)*. Cambridge: Zator Co.
- Stoffers, D., Berendse, H. W., Deijen, J. B., & Wolters, E. C. (2001). Motor perseveration is an early sign of Parkinson's disease. *Neurology*, 57, 2111–2113.
- Strenge, H., Lesmana, C. B., & Suryani, L. K. (2009). Random number generation in bilingual Balinese and German students: Preliminary findings from an exploratory cross-cultural study. *Perceptual and Motor Skills*, 109, 61–75.
- Taylor, K. I., Salmon, D. P., Monsch, A. U., & Brugger, P. (2005). Semantic and phonemic effects in random word generation: A dissociation between Alzheimer's and Huntington's disease patients. *Journal of the International Neuropsychological Society*, 11(3), 303–310.
- Towse, J. N., & Cheshire, A. (2007). Random number generation and working memory. *European Journal of Cognitive Psychology*, 19(3), 374–394.
- Tubau, E., & López-Moliner, J. (2009). Knowing what to respond in the future does not cancel the influence of past events. *PLoS ONE*, 4(5), e5607.
- Turing, A. (1936). On computable numbers, with an application to the entscheidungsproblem. *Proceedings of the London Mathematical Society*, 2(42), 230–265.
- Tversky, A., & Kahneman, D. (1971). Belief in the Òlaw of small numbers. *Psychological Bulletin*, 76, 105–110.
- Vandewiele, M., D'Hondt, W., Didillon, W., Iwawaki, S., & Mwamwendat, T. (1986). Number and color preferences in four countries. *Perceptual and Motor Skills*, 63(2), 945–946.
- Zvonkin, A., & Levin, L. (1970). The complexity of finite objects and the algorithmic concepts of information and randomness. *UMN = Russian Mathematic Surveys*, 25(6), 83–124.